3.12.1 Multi-dimensional Ito formula

Similarly the higher dimensional Ito formula is necessary in the case of a multi-dimensional Ito process. The following process is an example of a multi-dimensional Ito process,

$$\begin{cases} dX_1 &= u_1 dt + v_{11} dB_1 + \cdots + v_{1m} dB_m \\ \vdots &= \vdots \\ dX_n &= u_n dt + v_{n1} dB_1 + \cdots + v_{nm} dB_m \end{cases}$$

where each of the B_i's are one-dimensional Brownian motions and the above is essentially an n-dimensional Ito process. We can write the above in matrix notation as follows,

$$d\bar{X}(t) = \bar{u}dt + \bar{v}d\bar{B}(t)$$

Based on this process the multi-dimensional Ito formula is,

Theorem 19. *Suppose that $\bar{g}(t,x) = (g_1(t,x), \ldots, g_p(t,x))$ is twice continuous. Then the process*

$$\bar{Y}(t) = \bar{g}(t, \bar{X}(t))$$

is again an Ito process for which the following formula holds for each component k,

$$dY_k(t) = \frac{\partial g_k}{\partial t}(t, \bar{X}(t))dt + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(t, \bar{X}(t))dX_i(t) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, \bar{X}(t))dX_i(t)dX_j(t)$$

where

$$dB_i dB_j = \delta_{ij} dt, \quad \textit{and} \quad dB_i dt = dt dB_i = 0.$$

Example: (Brownian motion on the unit sphere). Use Ito's formula to write the following stochastic process,

$$\bar{X}(t) = (\cos B(t), \sin B(t))$$

in standard form

$$d\bar{X}(t) = u(t)dt + v(t)dB(t).$$

where $B(t)$ is considered to be 1-dimensional Brownian motion.

As usual one of our main task will be to define the function $g(t,x)$. Here $g(t,x)$ will in fact be a vector instead of a single function. We therefore define, in vector notation,

$$\bar{Y}(t) = \bar{g}(t,x) = e^{ix} = (\cos x, \sin x)$$

or written out in each component as,

$$\begin{aligned} Y_1(t) &= g_1(t,x) = \cos(x) \\ Y_2(t) &= g_2(t,x) = \sin(x) \end{aligned}$$

where we choose $x = B(t)$ and $B(t)$ denotes the usual one-dimensional Brownian motion. Using the multi-dimensional Ito formula from Theorem 19 above we obtain, first for g_1,

$$dY_1 = -\sin B(t)dB(t) - \frac{1}{2}\cos B(t)dt. \tag{3.39}$$

Similarly for g_2 we have using Ito formula,

$$dY_2 = \cos B(t) dB(t) - \frac{1}{2} \sin B(t) dt. \tag{3.40}$$

Using vector notation and the fact that $\bar{X}(t) = \bar{Y}(t) = (Y_1(t), Y_2(t)) = (\cos B(t), \sin B(t))$ we could write both of the above formulations (3.39, 3.40) as follows,

$$d\bar{X}(t) = K\bar{X}(t)dB(t) - \frac{1}{2}\bar{X}(t)dt \quad \text{where } K \text{ is the matrix,} \quad K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$