3.12 Ito Integration

In the previous section we made use of a so called "Ito formula",

$$d(\ln N(t)) = \frac{dN(t)}{N(t)} - \frac{1}{2}dt$$

which looks like a different way of differentiating the function $g(t,x) = \ln x$. We should not view it us such however. In fact we have no available differentiation theory for what we are about to do. Our calculus will depend solely on the integration theory parts of which we will examine below.

The Ito formula is an integral version of our well known chain rule. It is a formula which will allow us to evaluate our stochastic integrals. We start our presentation by providing the 1 dimensional Ito formula below, without proof, for any function

$$Y(t) = g(t, X(t)).$$

where $X(t)$ and $Y(t)$ are considered to be Ito processes.

But first we must explain what is an Ito process. An Ito process is nothing more than a *stochastic integral.* In other words, since integrals are essentially sums, an Ito process $X(s)$ can be thought of as sums of integrals of the following form,

$$X(s) = X(0) + \int_0^s u(t)dt + \int_0^s v(t)dB(t) \tag{3.38}$$

In other words, based on the above, an Ito process $X(t)$ is a family of integrals of the form (3.38) in terms of dt and the Brownian motion $dB(t)$. It is not unusual however to also write the Ito process $X(s)$ above in its differential form,

$$dX(t) = u(t)dt + v(t)dB(t).$$

Theorem 17. (The Ito Formula)
Suppose that $X(t)$ is an Ito process given by

$$dX(t) = udt + vdB(t)$$

Since $X(t)$ is an Ito process then the new function $Y(t) = g(t, X(t))$ is also an Ito process and further the following 1 dimensional Ito formula holds,

$$\boxed{dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2}$$

where $(dX(t))^2 = dX(t)dX(t)$ is computed based on the following rules,

$$dtdt = dtdB(t) = dB(t)dt = 0, \quad \textit{and} \quad dB(t)dB(t) = dt.$$

For the proof we refer to the book by Oksendal or Karantzas.

Let us take a quick look at some examples which can clarify how to use the Ito formula in order to integrate stochastic processes. Let us in fact start with the following simple example:

Example 1: Integrate

$$\int_0^s B(t)\, dB(t)$$

Let us choose $X(t) = B(t)$ and therefore the function g is,

$$g(t, x) = \frac{1}{2}x^2.$$

Thus, $Y(t) = g(t, B(t)) = \frac{1}{2}B^2(t)$, and the corresponding Ito formula reads,

$$dY(t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB(t))^2 = 0 + B(t)dB(t) + \frac{1}{2}dt$$

Remember that $Y(t) = \frac{1}{2}B^2(t)$. Thus the above can be written as,

$$d\left(\frac{1}{2}B^2(t)\right) = B(t)dB(t) + \frac{1}{2}dt$$

Rewriting this and integrating we obtain the answer for our original integral,

$$\int_0^s B(t)dB(t) = \int_0^s \frac{1}{2}B^2(t) - \int_0^s \frac{1}{2}dt = \frac{1}{2}B^2(s) - \frac{1}{2}s$$

Thus the answer to this integral is slightly different than what we would get if we had applied usual integration. There is a correction term of $-s/2$ involved which is attributed to the fact that we are using stochastic calculus. Sometimes the results from usual calculus and stochastic calculus integrals are the same but most of the time they are not. Whenever you integrate a stochastic process you should always use stochastic calculus just in case there is a correction term such as the one we found on this example.

Let us now look at a different example. Although we brushed it aside quickly during our SDE example we in fact looked at the integral of

$$\int_0^s dB(t) = B(s)$$

where $B(t)$ is our Brownian motion. Let us now revisit this integral and find out whether we did this correctly.

Example 2: Integrate

$$\int_0^s dB(t).$$

Here we choose $X(t) = B(t)$ and let the function g be,

$$g(t, x) = x.$$

Thus, $Y(t) = g(t, B(t)) = B(t)$, and the corresponding Ito formula becomes,

$$dY(t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB(t))^2 = 0 + 1dB(t) + 0$$

Since $Y(t) = B(t)$ then the above can be written as,

$$dB(t) = 1dB(t)$$

or integrating both

$$\int_0^s dB(t) = \int_0^s dB(t)$$

In other words there is no correction term using our Ito calculus. This therefore gives,

$$\int_0^s dB(t) = B(s)$$

as we used in our SDE example.

Example 3: Integrate

$$\int_0^s t\, dB(t).$$

Here we let $X(t) = B(t)$ and let the function g be,

$$g(t, x) = tx.$$

Thus, $Y(t) = g(t, B(t)) = tB(t)$, and the Ito formula gives,

$$dY(t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB(t))^2 = B(t)dt + tdB(t) + 0$$

Since $Y(t) = tB(t)$ the above can be written as,

$$d(tB(t)) = B(t)dt + tdB(t)$$

Integrating both sides and rearranging algebraically gives,

$$\int_0^s tdB(t) = sB(s) - \int_0^s B(t)\, dt$$

Note that we can not really integrate $\int_0^s B(t)\, dt$ above with respect to t since $B(t)$ is not a known function. Also note that the above is essentially similar to an integration by parts law but for stochastic processes this time.

Based on the last example we now present, again without proof, the general integration by parts formula.

Theorem 18. *Suppose that the following bounded and continuous function is given $f(s)$ which only depends on s. Then the following holds,*

$$\int_0^s f(t)\, dB(t) = f(s)B(s) - \int_0^s B(t)\, df(t)$$

As you may have noticed in this section we presented the 1 dimensional Ito formula. In fact a multi-dimensional version of it also exists.