ADVANCED EXAM

ALGEBRA

- **Groups:** Direct sums and products, Lagrange's Theorem, normal subgroups and quotient groups, homomorphism theorems, group actions on sets, conjugacy classes, centralizers and normalizers, Sylow theorems, center, commutators, nilpotent and solvable groups, simple groups, Jordan–Hölder and Schreier Theorems, finitely generated abelian groups.

- **Rings:** Ideals and residue class rings, homomorphism theorems, integral domains and fields of fractions, division rings, matrix rings, factorization theory (Euclidean domains, PID's, and UFD's), irreducibility criteria for polynomials, polynomial rings over UFD's, prime and maximal ideals in commutative rings, Noetherian rings, Hilbert basis theorem.

- **Modules:** Cyclic modules, torsion modules, free modules, direct sums, finitely generated modules over PID's, application to rational and Jordan canonical forms, bilinear forms, tensor products.

- **Fields:** Prime field, characteristic, finite fields, algebraic and transcendental extensions, algebraic closure, normal extensions, separable extensions, basic Galois theory, solvability by radicals, elementary symmetric polynomials.

REFERENCES

Allenby, *Rings, Fields and Groups*
Artin, *Algebra*
Artin, *Galois Theory*
Birkhoff & MacLane, *A Survey of Modern Algebra*
Dummit & Foote, *Abstract Algebra*
Hartley & Hawkes, *Rings, Modules and Linear Algebra*
Herstein, *Topics in Algebra*
Jacobson, *Basic Algebra*
Lang, *Algebra*
Stewart, *Galois Theory*
van der Waerden, *Algebra*

ANALYSIS

- Functions of bounded variation and the Riemann–Stieltjes integral.
- Borel and Lebesgue measure on the real line; upper and lower semi-continuity; Borel and Lebesgue measurable functions; the Lebesgue integral on the line; comparison of the Lebesgue and Riemann integrals; convergence almost everywhere and convergence in measure; Fatou's lemma; monotone, bounded and dominated convergence theorems; Egorov's Theorem and Lusin's Theorem.
- Lebesgue integral in $\mathbf{R}^n$; Fubini's Theorem.
- Differentiation of monotone functions, differentiation of integrals, absolutely continuous and singular functions; the Radon–Nikodym Theorem.
- Convex functions; inequalities of Schwarz, Hölder, Minkowski, and Jensen.
- L_p-spaces and their duals; Riesz–Fischer Theorem; modes of convergence.
- Elementary Hilbert space theory, orthonormal bases, Bessel's inequality and Parseval's identity; mean-square convergence of Fourier series; Riemann–Lebesgue Lemma.
- General theory of measure and integration; signed measures and the Hahn decomposition; absolute continuity and the Lebesgue decomposition; the Radon–Nikodym Theorem; outer measures and Lebesgue–Stieljes measure; the Carathéodory–Hahn extension theorem; product measures.

REFERENCES

Berberian, *Introduction to Hilbert Spaces*
Gelbaum and Olmsted, *Counterexamples in Analysis*
Halmos, *Measure Theory*
Royden, *Real Analysis*
Rudin, *Principles of Mathematical Analysis*
Rudin, *Real and Complex Analysis*
Wheeden and Zygmund, *Measure and Integral*

DIFFERENTIAL EQUATIONS

- Constant coefficient linear systems of ODE, normal forms, exponential matrix solutions, variation of parameters formula.

- Well-posedness of the initial-value problem: local existence, uniqueness and continuous dependence for nonlinear systems of ODE; continuation and global existence; Picard's (iteration) method and Euler's (finite difference) method; Gronwall's inequality.

- Limit sets and invariant sets: equilibria, limit cycles, ω and α-limit sets; invariant manifolds (stable, unstable, center).

- Stability theory: linearization at an equilibrium point, Lyapunov stability.

- Two-dimensional systems (plane autonomous systems): phase portraits, Poincaré–Bendixson theory; qualitative analysis of special systems, such as gradient or Hamiltonian systems.

- Elementary facts about distributions: weak derivatives and mollifiers, convolutions and the Fourier transform.

- The prototype linear equations of hyperbolic, elliptic and parabolic type: the wave, potential (Laplace/Poisson), and diffusion (heat) equations; fundamental solutions for these equations.

- Initial-value problems for PDE: the hyperbolic Cauchy problem (wave equation), basic features of wave propagation (D'Alembert's solution and Kirchhoff's solution), characteristics, energy estimates; the parabolic Cauchy problem (diffusion equation), basic features of diffusion phenomena.

- Boundary-value problems for elliptic PDE: the Laplace and Poisson equations with Dirichlet, Neumann or periodic boundary conditions; variational formulation and weak solutions, the Sobolev spaces H^1 and H_0^1; the associated eigenvalue problem and eigenfunction expansions; Green's functions; the maximum principle.

- Mixed initial/boundary-value problems: separation of variables and eigenfunction expansion methods.

REFERENCES

M. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*

L. Perko, *Differential Equations and Dynamical Systems*

J. Hale, *Ordinary Differential Equations*

P. Garabedian, *Partial Differential Equations*

F. John, *Partial Differential Equations*

I. Stakgold, *Boundary Value Problems of Mathematical Physics, I and II*

F. Treves, *Basic Linear Partial Differential Equations*

V.S. Vladimirov, *Equations of Mathematical Physics*

GEOMETRY

- Inverse and implicit function theorems, rank of a map. Regular and critical values. Sard's theorem.
- Differentiable manifolds, submanifolds, embeddings, immersions and diffeomorphisms.
- Tangent space and bundle, differential of a map. Partitions of unity, orientation, transversality, embeddings in $\mathbf{R}^n$.
- Vector fields, local flows, Lie bracket, Frobenius theorem.
- Lie groups (generalities), matrix Lie groups, left-invariant vector fields, Lie algebra of a Lie group.
- Tensor algebra, tensor fields, differential forms, the exterior differential, integration, Stokes theorem, closed and exact forms, deRham's cohomology.
- Vector bundles, normal bundle. Operations on vector bundles.
- Surfaces in $\mathbf{R}^3$. Gaussian and mean curvature. The Theorema Egregium.
- Rudiments of Riemannian geometry. The Riemannian (Levi–Civita) connection. The curvature tensor.

REFERENCES

Auslander and Mackenzie, *Introduction to Differentiable Manifolds*
Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*
do Carmo, *Differential Geometry of Curves and Surfaces*
Guillemin and Pollack, *Differential Topology*
Morgan, *Riemannian Geometry*
Spivak, *Differential Geometry, Vol. I, II*
Warner, *Foundations of Differentiable Manifolds and Lie Groups*

MATHEMATICAL STATISTICS

- Probability models in a measure-theoretic setting, random variables, mathematical expectation (based on abstract Lebesgue integral), conditional expectation (with respect to a sigma-field), modes of convergence of random variables, characteristic functions, independence, limit theorems (weak and strong laws of large numbers, central limit theorem), martingales.

- Sampling distributions, sufficiency, completeness, exponential families.

- Parametric estimation, maximum likelihood estimation, asymptotic theory, consistency, efficiency, admissibility, Bayes estimators, Cramer–Rao inequality, Rao–Blackwell and Lehmann–Scheffe theorems, confidence intervals and sets.

- Hypothesis testing, Neyman–Pearson lemma, most powerful tests, optimality criteria, likelihood ratio tests, asymptotic properties of tests.

REFERENCES

Shiryayev, *Probability*
Durrett, *Probability: Theory and Examples*
Rao, *Linear Statistical Inference and Its Applications*, 2nd ed.
Lehmann, *Theory of Point Estimation*
Lehmann, *Testing Statistical Hypotheses*, 2nd ed.
Arnold, *Mathematical Statistics*

LINEAR MODELS: THEORY AND APPLICATION

This exam covers the theory and application of the linear model based mostly on the course content of Stat 705–706. Students should be able to apply the results to data and understand issues in modelling (though the length of the exam prohibits much data analysis.) There will not be questions on matrix theory alone, but students are expected to have a working knowledge of necessary matrix results as used in Stat 705–706 (nonnegative matrices, idempotent matrices, generalized inverses, etc.).

a) Random vectors; multivariate normal; properties of linear and quadratic forms; noncentral chi-square, F- and t-distributions.

b) General theory of estimation, testing and confidence intervals in the linear model of full or less than full rank; least squares and maximum likelihood estimation, F-tests, methods of simultaneous inference, power considerations, estimability, incorporating linear restrictions.

Problems arising in

i) Regression. General inferences in the multiple regression model, fixed and random regressors, correlation (simple, partial, multiple), prediction intervals, properties of residuals and residual analysis, calibration/inverse-prediction, weighted least squares.

ii) Experimental Design.

Inference in balanced or unbalanced complete multifactor design; simultaneous tests and confidence intervals; estimability; randomized block designs; incomplete designs including BIBD's and Latin squares; analysis of covariance; random effects, variance component and mixed models (balanced one factor random effects models, balanced two factor random effects and mixed models, two factor nested models and balanced model with crossing and nesting).

REFERENCES

Draper and Smith, *Applied Regression Analysis*, 2nd ed.
Montgomery, *Design and Analysis of Experiments*, 2nd ed.
Graybill, *Theory and Application of the Linear Model*
Seber, *Linear Regression Analysis*
Searle, *Linear Models*
Scheffé, *The Analysis of Variance*
Arnold, *The Theory of Linear Models and Multivariate Analysis*