Math 412/512H Final Exam (Take Home) Spring 2008

Hand in your solutions on or before May 19.

Name: ______________________________

1. (16)	________
2. (16)	________
3. (16)	________
4. (16)	________
5. (16)	________
6. (20)	________
TOTAL (100)	________

1. Let F be a finite field of characteristic p. Let $\phi : F \to F, \phi(a) = a^p$ be the Frobenius map. Prove that

   (a) ϕ is an automorphism of F.

   (b) $\phi^n(a) = a^{p^n}$ for all $a \in F$.

   (c) The set $K = \{a \in F \mid \phi^n(a) = a\}$ is a subfield of F.

2. Prove that every algebraically closed field is infinite.

3. Let F be any field. Consider the statements

   (a) F is algebraically closed.

   (b) $f(x) \in F[x]$ is irreducible if and only if $\deg(f) = 1$.

   (c) Every nonconstant polynomial in $F[x]$ splits over F.

   Prove that (a) implies (b) and that (b) implies (c).

4. (a) Let $\zeta = e^{2\pi i/6}$ be a primitive 6-th root of unity. Compute the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.
   (b) Let $E = \mathbb{Q}(\sqrt{3}, i)$. Compute the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.

5. Let R be a ring with unity $1 \in R$ and let $n = \text{char} R$. Define $\phi : \mathbb{Z} \to R$ by

$$\phi(k) = \begin{cases} \underbrace{1 + 1 + \ldots + 1}_{k \text{ times}} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ \underbrace{-1 - 1 - \ldots - 1}_{|k| \text{ times}} & \text{if } k < 0 \end{cases}$$

(a) Prove that ϕ is a ring homomorphism.

(b) Prove that $\ker \phi = n\mathbb{Z}$.

Remark: This explains the terminology *characteristic zero*.

6. Let R and S be rings and let $\phi : R[x] \to S$ be the ring homomorphism defined by $\phi(f(x)) = f(0)$.

   (a) Prove that if $R = S$ then $\ker\, \phi = <x>$.

   (b) If $R = \mathbb{Z}$ and $S = \mathbb{Z}_p$ then let $I \subset R[x]$ be the subset of all polynomials whose constant term is a multiple of p. Prove that $\ker\, \phi = I$.

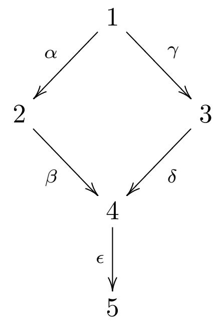
   (c) Prove that the set I in (b) is an ideal in $\mathbb{Z}[x]$ and find an element $f \in \mathbb{Z}[x]$ such that $I = <f>$.

   (d) Conclude that $\mathbb{Z}_p$ is a quotient ring of $\mathbb{Z}[x]$.

MATH 512 students only

7. You can choose to do this problem instead of problem # 6. If you work on both problems you must indicate which one should be considered for your grade.

   Let Q be the quiver



   (a) Find the dimension vectors of the five indecomposable projective representations P_1, P_2, P_3, P_4, P_5 of Q.

   (b) Give the linear maps $\varphi_\alpha, \varphi_\beta, \varphi_\gamma, \varphi_\delta, \varphi_\epsilon$ of P_1 and P_4 in terms of matrices.

   (c) Show that $\mathrm{Hom}(P_4, P_1)$ is of dimension 2.