Math 545 Advanced Linear Algebra - Midterm

Show your work. A correct answer without explanation will receive little credit.

1. Let V be a finite dimensional vector space over field F. $\{v_1, \ldots, v_n\}$ is a set of basis for V. Take another set of vectors $\{u_1, \ldots, u_n\}$, where

$$u_i = \sum_{j=1}^{n} \alpha_{ij}\ v_j \qquad i = 1, \ldots, n.$$

   (a) (15 points) Prove that $\{u_1, \ldots, u_n\}$ is also a basis for V if and only if the matrix $A = (\alpha_{ij})$ is invertible.

   (b) (Extra 15 points) Suppose V also has an inner product, and $(v_1, \ldots, v_n)$ is an orthonormal basis. Prove that $\{u_1, \ldots, u_n\}$ is also an orthonormal basis if and only if the matrix $A = (\alpha_{ij})$ is an orthogonal matrix.

(a) **Proof:** Clearly, $\dim(V) = n$. So $\{u_1, \ldots, u_n\}$ is a basis if and only if they are linearly independent, i.e., the equation

$$\beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_n u_n = 0 \tag{1}$$

has only zero solutions: $\beta_1 = \ldots = \beta_n = 0$. Rewrite Eq. (1) as

$$\sum_{k=1}^{n} \beta_k\ u_k = \sum_{k=1}^{n} \beta_k \left(\sum_{j=1}^{n} \alpha_{kj} v_j \right) = \sum_{j=1}^{n} v_j \left(\sum_{k=1}^{n} \beta_k \alpha_{kj} \right) = 0,$$

which is equivalent to (because $\{v_1, \ldots, v_n\}$ is a basis)

$$\sum_{k=1}^{n} \beta_k \alpha_{kj} = 0, \qquad j = 1, \ldots, n, \qquad \text{or} \quad A'\vec{\beta} = 0,$$

where $\vec{\beta}$ is the column vector consisting of $\beta_1, \ldots, \beta_n$. Clearly, the above equation has only zero solutions if and only if A is invertible.

(b) **Proof:** Consider the inner product of u_i and u_j:

$$(u_i, u_j) = \left(\sum_{k=1}^{n} \alpha_{ik} v_k, \sum_{l=1}^{n} \alpha_{jl} v_l \right) = \sum_{k=1}^{n} \alpha_{ik} \alpha_{jk}, \qquad i, j = 1, \ldots, n,$$

as $\{v_1, \ldots, v_n\}$ is an orthonormal basis ($(v_i, v_j) = 1$ if $i = j$; $(v_i, v_j) = 0,$ if $i \neq j$),

Let the matrix $U = (u_{ij})$, where $u_{ij} = (u_i, u_j)$. Then $U = AA'$, consequently $\{u_1, \ldots, u_n\}$ is an orthonormal basis if and only if the matrix $A = (\alpha_{ij})$ is an orthogonal matrix.

2. Given four vectors in $\mathbb{R}^4$: $v_1 = (1, 0, 2, 2)$, $v_2 = (1, 1, 0, 4)$, $v_3 = (0, 1, 2, -1)$, $v_4 = (0, 0, 4 - 3)$. Let V be the subspace of $\mathbb{R}^4$ spanned by v_1, v_2, v_3, v_4.

   (a) (7 points) Find the dimension of V;

   (b) (5 points) Find a basis of V;

   (c) (13 points) Find an orthonormal basis for V.

   Solution: Reduce the following matrix into echelon form:

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 4 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Dim(V) = 3.

   (b) A basis of V: $(1, 0, 2, 2), (0, 1, -2, 2), (0, 0, 4, -3)$. (Non-zero rows of the final matrix in echelon form.)

   (c) Gram-Schmidt process ($w = w_r - \sum_{k=1}^{r-1} u_k(w_r, u_k)$ and $u_r = w/\|w\|$):

   First step: $w_1 = (1, 0, 2, 2), \|w_1\| = 3 \to u_1 = (1/3, 0, 2/3, 2/3)$.

   Second step:$w_2 = (0, 1, -2, 2), (w2, u_1) = 0, \to w = w_2, \|w\| = 3$. So $u_2 = (0, 1/3, -2/3, 2/3)$.

   Last step:: $w_3 = (0, 0, 4, -3), (w_3, u_1) = 2/3, (w_3, u_2) = -14/3$, So $w = w_3 - 2/3 * u_1 + 14/3 * u_2 = (-2/9, 14/9, 4/9, -3/9)$. $\|w\| = 5/3$. So $u_3 = (-2/15, 14/15, 4/15, -1/5)$.

   One orthonormal basis: $u_1 = (1/3, 0, 2/3, 2/3), u_2 = (0, 1/3, -2/3, 2/3), u_3 = (-2/15, 14/15, 4/15, -1/5)$.

3. (15 points) Determine whether there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, satisfying $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$. If yes, give an example; if no, prove it.

   Solution: Yes, such linear transformation does exist because $(1, -1, 1)$ and $(1, 1, 1)$ are linearly independent vectors in $\mathbb{R}^3$.

   To completely define a linear transformation, it suffices to define the images for a basis. So we can define a linear transformation T with $T(1, -1, 1) = (1, 0)$, $T(1, 1, 1) = (0, 1)$, and $T(1, 0, 0) = (1, 1)$. Note that (1,-1,1), (1,1,1), and (1,0,0) are a basis of $\mathbb{R}^3$.

4. Let $T \in \mathcal{L}(V, V)$, where V is a finite dimensional vector space over field F. Suppose $T^2 = T$.

   (a) (10 points) Prove that $T(V) \cap N(T) = \{0\}$, i.e., the intersection of the range space and null space has only the zero vector.

   (b) (10 points) Prove that $V = T(V) + N(T)$.

   (c) (5 points) Compute $T(V)$ and $N(T)$ when T is the identity transformation ($I^2 = I$).

   (a) **Proof:** For any $v \in T(V) \cap N(T)$. we have $T(v) = 0$, and there exists $w \in V$ such that $v = T(w)$. Thus $0 = T(v) = T^2(w) = T(w) = v$.

   (b) **Proof:** According to the fundamental theorem: $\dim(T(V)) + \dim(N(T)) = \dim(V)$. So $V = T(V) + N(T)$.

(Details: Denote the dimension of V and $T(V)$ as n and m respectively. Let $\{v_1, \ldots, v_m\}$ be a basis of $T(V)$, and $\{v_{m+1}, \ldots, v_n\}$ be a basis of $N(T)$. If

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = 0,$$

then

$$\alpha_1 v_1 + \ldots + \alpha_m v_m = -(\alpha_{m+1} v_{m+1} + \ldots + \alpha_n v_n) \in (T(V) \cap N(T)).$$

So

$$0 = \alpha_1 v_1 + \ldots + \alpha_m v_m = -(\alpha_{m+1} v_{m+1} + \ldots + \alpha_n v_n),$$

from which we know $\alpha_1 = \ldots = \alpha_n = 0$. Therefore $v_1, \ldots, v_n$ are linearly independent, thus a basis of V. So $V = T(V) + N(T)$.)

(c) **Solution:** $T(V) = V, N(T) = \{0\}$

5. (20 points) Compute the minimal polynomial for the following matrices:

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Solution: Use the formula given in the textbook.

a) For $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, $f(x) = (x-1)(x-2)$. So the minimal polynomial $m(x)$ could be $(x-1)$, $(x-2)$ or $(x-1)(x-2)$. Plugging the matrix into these three polynomials, one can show that $m(x) = (x-1)(x-2)$.

b) Same procedure as in a), $m(x) = (x-1)(x-2)$.

c) $m(x) = (x-2)^2$.

d) $m(x) = (x-2)$.