The existence of regular weak solutions for the equations of motion of 3D compressible, viscous fluid flows with the no-slip boundary conditions

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Abstract

We consider the equations of motion of a compressible, viscous, isentropic fluid in a bounded domain of $\mathbb{R}^3$ with the no-slip boundary conditions. Given a constant, equilibrium state $(\overline{\rho}, \mathbf{0})$, $\overline{\rho} > 0$, we construct a global in time, regular weak solution, provided that initial data $(\rho_0, \mathbf{u}_0)$ are close to the equilibrium when measured by $|\rho_0 - \overline{\rho}|_{L^\infty} + |\mathbf{u}_0|_{W^{1,2}}$ and discontinuities of ρ_0 decay near the boundary.

Contents

1 Introduction

We consider the Navier-Stokes equations that describe the motion of a compressible, isentropic, viscous flow with $\rho(t,x)$ and $\mathbf{u}(t,x)$ being the density and the velocity of the fluid.

$$\frac{\partial}{\partial t}\rho + \text{div }(\rho\mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \text{div }(\rho\mathbf{u}\otimes\mathbf{u}) - (\lambda+\mu)\text{div }\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = \rho\mathbf{f}, \tag{2}$$

$$(t,\, x) \in \mathbb{R}_+ \times \Omega, \quad P(\rho) = A\rho^\gamma,\, A > 0,\, \gamma \geq 1. \tag{3}$$

Let

$$3\lambda + 2\mu \geq 0, \quad \mu > 0, \tag{4}$$

and a set of initial and boundary conditions be specified.

$$(\rho(0,x),\, \mathbf{u}(0,x)) = (\rho_0(x),\, \mathbf{u}_0(x))\,, \quad x \in \Omega, \tag{5}$$

$$\mathbf{u}(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega. \tag{6}$$

Alternatively, other types of problems can be considered. For example, (a) periodic flows; functions are assumed to be spatially periodic in each direction, (b) Cauchy problem; $\Omega = \mathbb{R}^3$ and $\mathbf{u}$ is prescribed a value at $x = \infty$.

There is an extensive literature concerning the solvability of these problems. For the detailed discussion of the results we refer the reader to the monograph NOVOTNÝ et al.[11].

If the data of the problem are smooth, say $(\rho_0, \mathbf{u}_0) \in W^{1,q}(\Omega) \times \left(W^{2,q}(\Omega)\right)^3$, $q > 3$, the initial-boundary value problem is known to be well-posed, meaning that there is a time interval on which the solution exists and retains its initial smoothness, SOLONIKOV[12].

On the other hand there is a well-developed theory of weak solutions, see P.-L.LIONS[6], FEIREISL[3]. The typical result is the following.

Theorem (P.-L. Lions, [6]). *Suppose that $\gamma \geq \frac{9}{5}$ and $\Omega \in C^{2+\theta}$, $\theta > 0$. Suppose that the initial data (ρ_0, m_0) satisfy $\rho \in L^\gamma(\Omega)$, $|m_0|^2/\rho_0 \in L^1(\Omega)$, where we agree that $m_0 = 0$ on $\{\rho_0(\cdot) = 0\}$. Then there is a global weak solution of the problem (1)–(6), $(\rho, \mathbf{u})$, such that $\rho(0,\cdot) = \rho_0(\cdot)$ and $\rho(0,\cdot)\mathbf{u}(t,\cdot) = m_0$ Moreover, for any $t > 0$ the* *energy inequality holds.*

$$\int_\Omega \left(\frac{1}{2}\rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2 + \frac{A\rho(t,\cdot)^\gamma}{\gamma-1}\right) + \int_0^t \int_\Omega \mu|\nabla \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\,\mathbf{u}|^2 \leq \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}_0|^2 + \frac{A\rho_0^\gamma}{\gamma-1}\right). \quad (7)$$

Solutions constructed in the above theorem have minimal regularity properties; $\rho \in L^\infty(\mathbb{R}_+ : L^\gamma(\Omega))$ and $\mathbf{u} \in L^2\left(\mathbb{R}_+ : W^{1,2}(\Omega)\right)^3$.

For the Cauchy problem, $\Omega = \mathbb{R}^3$, the global existence of a weak solution which stays "close" to the equilibrium sate, $(\bar{\rho}, 0)$, was proven in HOFF[4]. The important feature, in contrast with the result of SOLONIKOV [12], is that the solution is essentially weak; the density is an L^∞ function. On the other hand, the solution possess many favorable properties, for example, the absence of vacuum, which is, in fact, necessary to justify the use of the equations (1) and (2) as a model of motion of fluids.

Theorem (D.Hoff, [4]). *Let $\bar{\rho} > 0$ and $L > 0$ be given. There are positive constants a and C, depending on μ, λ, A, L and there is a global positive constant θ, such that, given initial data $(\rho_0, \mathbf{u}_0)$ satisfying*

$$|\rho - \bar{\rho}|^2_{L^\infty(\mathbb{R}^3)} + \int_{\mathbb{R}^3} |\mathbf{u}_0(y)|^2 + (\rho_0(y) - \bar{\rho})^2 \leq a$$

and

$$|\mathbf{u}_0|_{L^6(\mathbb{R}^3)^3} \leq L,$$

there is a global weak solution $(\rho, \mathbf{u})$ of the problem (1)–(5) for which

$$|\rho - \bar{\rho}|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)} \leq C C_0^\theta,$$

$$|\mathbf{u}(t,\cdot)|_{L^\infty(\mathbb{R})^3} \leq C(\tau) C_0^\theta, \quad t \geq \tau > 0.$$

(We refer the reader to [4] for the complete statement of the Theorem.)

In this work we present a development of the existence theory of the near equilibrium weak solution presented in HOFF[4] that accommodates the presence of no-slip boundaries. The density component of the weak solution that we construct is *essentially* L^∞ away form the boundaries and is such that discontinuities in $\rho(t,\cdot)$ decay near the boundary at the fixed rate. Specifically, we measure $\rho(t,\cdot)$ by the following seminorm.

$$\langle \rho(t,\cdot)\rangle_{\alpha,\infty} = \sup_{x\in\partial\Omega} \operatorname{ess\,sup}_{y\in\Omega} \frac{|\rho(t,x) - \rho(t,y)|}{|x-y|^\alpha}, \quad \alpha \in]0,1[. \quad (8)$$

This "localization" of discontinuities in $\rho(t,\cdot)$ corresponds to a physical situation when motion of a fluid results from disturbances that occur in the interior of the domain. At the level of technical description of the proof, the introduction of the above functional to measure the strength of sound waves (density), instead of just $|\rho(t,\cdot)|_{L^\infty}$, is dictated by the fact that the strength of the wave reflected by the boundary, as measured by the L^∞ norm, can not be bounded by the strength of the incident wave, measured by the same norm. Moreover, for the solution to remain near the equilibrium state we also impose a certain structural restriction on the model (relative size of λ and μ, given by (11)) which guarantees that the strength of the sound wave reflected by the boundary is, in fact, *smaller* than the strength of the incident wave.

1.1 Statement of the result

Consider the initial-boundary value problem (1)–(6) for unknown functions $\rho(t,x)$ and $\mathbf{u}(t,x)=(u_1(t,x),u_2(t,x),u_3(t,x))$. We assume that Ω is the unit ball, $\mathbf{f}=0$ and $\gamma=1$.

Definition 1. *A pair of functions*

$$(\rho(t,x),\ \mathbf{u}(t,x))\in L^1_{loc}\left(\mathbb{R}_+\times\Omega\right)\times L^2\left(\mathbb{R}_+;W_0^{1,2}\left(\Omega\right)^3\right)$$

is called a weak solution of (1)-(6) *if*

$$\rho u_i,\ \rho u_k\otimes u_l,\in L^1_{loc}\left(\mathbb{R}_+\times\Omega\right),\ i,k,l=1..3,$$

for all test functions $\phi\in C_0^\infty\left(\mathbb{R}\times\Omega\right),\ \psi\in C_0^\infty\left(\mathbb{R}\times\Omega\right)^3$, *it holds (summation over the repeated indexes is assumed)*

$$\int\int_{\mathbb{R}_+\times\Omega}\rho\partial_t\phi+\rho\mathbf{u}\cdot\nabla\phi=-\int_\Omega\rho_0\phi\big|_{t=0}, \tag{9}$$

$$\int\int_{\mathbb{R}_+\times\Omega}\rho u_k\partial_t\psi_k+\rho u_k u_j\partial_k\psi_j$$
$$-\int\int_{\mathbb{R}_+\times\Omega}(\lambda+\mu)\mathrm{div}\,\mathbf{u}\mathrm{div}\,\psi+\mu\partial_k u_l\partial_k\psi_l-P\mathrm{div}\,\psi=-\int_\Omega\rho_0 u_{0,k}\psi_k\big|_{t=0}. \tag{10}$$

Definition 2. *Define the space* $L^{\infty}_{\alpha,\partial\Omega}(\Omega)$, $\alpha \in]0,1[$, *to be the set of* $f \in L^{\infty}(\Omega)$ *such that* $\forall x \in \partial\Omega$, $\operatorname{ess}\lim_{y\to x} f(y) \triangleq f(x)$ *exists and* $\langle f\rangle_{\alpha,\infty}$ *is finite, where the seminorm* $\langle\cdot\rangle$ *is defined in* (8).

We prove the following theorem.

Theorem A. *For any* $\overline{\rho} > 0$, $L > 0$ *and* $\alpha \in]0, \frac{1}{2}[$ *there are* $c_0 = c_0(\alpha) > 0$ *and* $c_i = c_i(\overline{\rho}, \lambda, \alpha, \mu, A, L)$, $i = 1, 2$, *such that if*

$$\frac{\mu}{5\lambda + 7\mu} < c_0, \tag{11}$$

$$(\rho_0,\, \mathbf{u}_0) \in L^{\infty}_{\alpha,\partial\Omega}(\Omega) \times W^{1,2}_0(\Omega)^3, \quad \frac{1}{meas(\Omega)}\int_{\Omega} \rho_0 = \bar{\rho}, \tag{12}$$

$$\langle \rho_0 \rangle_{\alpha,\partial\Omega} < L \tag{13}$$

and

$$|\rho_0 - \bar{\rho}|_{L^{\infty}(\Omega)} + |\nabla \mathbf{u}_0|_{L^2(\Omega)^9} \leq c_1, \tag{14}$$

then there exists a weak solution of (1) - (6), $(\rho,\, \mathbf{u})$, *defined for all times* $t > 0$. *Moreover,*

$$(\rho,\, \mathbf{u}) \in L^{\infty}(\mathbb{R}_+ \times \Omega) \times L^{\infty}\left(\mathbb{R}_+\,;\, W^{1,2}_0(\Omega)^3\right), \tag{15}$$

and it verifies the following estimates.

$$\begin{aligned}
&\sup_{(0,+\infty)\times\Omega} |\rho(t,x) - \overline{\rho}| + \sup_{(0,+\infty)} \langle \rho(t,\cdot)\rangle_{\alpha,\partial\Omega} \leq c_2,\\
&\inf_{(0,+\infty)\times\Omega} \rho(t,x) \geq c_2^{-1},\\
&\sup_{t\in(0,+\infty)} \left\{ |\mathbf{u}(t,\cdot)|_{L^2(\Omega)} + |\nabla \mathbf{u}(t,\cdot)|_{L^2(\Omega)} \right\} \leq c_2,
\end{aligned}$$

for some $c_2 > 0$.

Some remarks should be mentioned here.

Remark 1. *The special domain* Ω, *the unit ball, is chosen here for the sole purpose of having a simple formulae of its Green function. The generalization to smooth domains* Ω *of the class* C^2 *is straightforward.*

Remark 2. *Since the solution, constructed in the Theorem, is such that the oscillations in density are small there is no loss of generality in assuming the pressure law* $P = A\rho$ *instead of the "isentropic"* γ*–law,* $P = A\rho^{\gamma}$, $\gamma \geq 1$. *Indeed, the derivation of a priori estimates in case* $\gamma > 1$ *is identical to case* $\gamma = 1$. *Moreover, the strong convergence of the sequence of the approximate, classical solutions for* $\gamma > 1$ *is established by the Lions-Feireisl theory, see FEIREISL[3].*

Remark 3. *Additional regularity properties for the constructed solution hold. For example, flow trajectories are Hölder continuous throughout the domain and Lipschitz continuous near the boundary.*

The framework of the analysis was established in works of HOFF[4] and P.L. LIONS[6]. We shortly describe the new issues brought to the problem by the presence of no-slip boundaries.

By means of the energy methods, one proves that if the thermodynamic parameters of the flow (in our case ρ) stay close to the equilibrium state (measured by $|\rho(t,\cdot)-\bar{\rho}|_{L^\infty}$) on the time interval $[0,T]$, then the kinetic parameter, $\mathbf{u}$, remains close to the zero equilibrium for all times in $[0,T]$, provided that it is close initially. This gives fair amount of regularity of the flow, for example, $L^2\left((0,T)\times\Omega\right)^3$ integrability of the acceleration $\rho\dot{\mathbf{u}}$. Then, the coupling mechanism between (1) and (2) is studied. To do this equations (2) are written in the form

$$-(\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u}-\mu\Delta\mathbf{u}=\,-\,\rho\dot{\mathbf{u}}-\nabla P. \tag{16}$$

This is the system of Lamé equations of the linear elasticity. Applying div operator to the above equations one gets the well known equation for the viscous flux $F=(\lambda+2\mu)\mathrm{div}\,\mathbf{u}-P$, see for example HOFF[4].

$$-\Delta F=\,-\,\mathrm{div}\,\rho\dot{\mathbf{u}}.$$

The right-hand side of the equation belongs to $W^{-1,2}(\Omega)$ and $F\in L^2(\Omega)$, by the energy estimates. From the elliptic regularity results it follows: $F\in W^{1,2}_{\mathrm{loc}}(\Omega)$, meaning that there is a gain in regularity of F in the interior of Ω as compared with regularity of $P\in L^1$ and $\mathrm{div}\,\mathbf{u}\in L^2$, obtained from the first energy inequality. In the case when $\Omega=\mathbb{R}^3$ or the periodic case the argument is global. Furthermore, using (formally) the inverse Laplace operator $(\Delta)^{-1}[\cdot]$ and writting

$$\begin{aligned}F=(\Delta)^{-1}[\mathrm{div}\,\rho\dot{\mathbf{u}}]&=(\Delta)^{-1}\dot{[}\mathrm{div}\,\rho\mathbf{u}]\\&+\left\{(\Delta)^{-1}[\mathrm{div}\,\mathrm{div}\,\rho\mathbf{u}\otimes\mathbf{u}]-\mathbf{u}\cdot\nabla(\Delta)^{-1}[\mathrm{div}\,\rho\mathbf{u}]\right\},\end{aligned} \tag{17}$$

one notices that the expression in curly brackets is a sum of commutators of Reisz transforms and operators of multiplication by u_i, $i=1..3$. A refined estimates for the viscous flux, ($F(t,\cdot)\in W^{1,q}$, $q>3$), are used to derive *a priori* uniform bounds on ρ by means of the equation of conservation of mass, with the notation for the derivative along a trajectory $\dot{\{...\}}=\frac{d}{dt}\{...\}$,

$$\dot{\log}\,\rho+(\lambda+2\mu)^{-1}P=\,-\,(\lambda+2\mu)^{-1}F.$$

This type of argument is insufficient in the presence of the no-slip boundaries since a) there is a new type of waves, reflected sound waves, that interact with the flow and b) the no-slip boundaries contribute additional terms to the expression for F, (17), that require the detailed study of their regularity properties.

The restriction of F to the boundary of the domain is the solution of a certain integral equation, called, in the elasticity theory, Liechtenstein equation. By using the integral representation of the normal stress, obtained form the corresponding representation of the solution of the Lamé equations (16), one notices the presence of following term in the formulae for F.

$$F(t,x) = \frac{\mu}{5\lambda + 7\mu} \int_{\Omega} \frac{\partial^2}{\partial_{x_i} \partial_{y_i}} K(x,y)(P(t,y) - P(t,x))\, dy + ..., \quad (18)$$

where $K(x,y) = (4\pi|y||x - y|y|^{-2}|)^{-1}$, $y \neq 0$. One sees that $x = y \in \partial\Omega$ is the singular set for the kernel of this integral and the corresponding singular integral operator, acting on P, does not map L^∞ to itself. In order to force the range of this operator to be a subset of L^∞ we assume that discontinuities in pressure (density) field decay as $y \to y_0 \in \partial\Omega$ and we use the functional defined in (8) to measure it.

There are two issues that come up immediately. First of all the values of the integral in (18), can be of an arbitrary sign and in order to control oscillations in the density this term must be compared with the damping effect of pressure. We are able to prove that the latter dominates if the ratio

$$\frac{\mu}{5\lambda + 7\mu} < c(\alpha),\ \alpha \in]0, \tfrac{1}{2}[,$$

for some $c(\alpha)$. Unfortunately, we were not able to verify that there is a value of α for which the above condition holds irrespectively of λ and μ in the range $\mu > 0$, $3\lambda + 2\mu \geq 0$. Secondly, we must be sure that the corresponding flow is regular enough to preserve the rate of the decay of discontinuities of ρ near the boundary. The analysis will be consistent only if we succeed to prove that trajectories of the flow are Lipschitz continuous, at least "near the boundary". Let us look at the representation formulae for the values of $\mathbf{u}$. It can be written in the following form.

$$\mathbf{u}(t,x) = \int G(x,y) \nabla_y P(t,y) + ...,$$

where $G(x, y)$ is the Green function of the domain Ω. The boundedness of P implies that $\mathbf{u}$ is quasi-Lipschitz continuous and thus, we expect that trajectories are Hölder continuous. On the other hand, since discontinuities in P vanish near the boundary, as measured by (8), it is possible to prove that $\mathbf{u}$ is Lipschitz continuous "near" the boundary.

By restricting the initial data in a suitable way we obtain *a priori* estimates for the solution (classical) in some strong norms. A weak solution, with the properties stated in the theorem, is constructed as a limit of the classical solutions, corresponding to the smoothed initial data.

2 A priori estimates

2.1 Functional setting

Let $B(x,r)$, $r > 0$, $x \in \mathbb{R}^3$, be the ball with radius r, centered at x. Let $L^p(\Omega)^3$, $1 \leq p \leq +\infty$, be the Lebesgue space of functions from domain Ω to $\mathbb{R}^3$, integrable with exponent p (essentially bounded when $p = +\infty$). We use $|\cdot|_p$ to denote the norm in these spaces. Let $W_0^{k,p}(\Omega)^n$, $k = 1, 2$, $1 \leq p < +\infty$ be the space of weakly differentiable, up to the order k, functions that have zero trace on the boundary $\partial\Omega$, with norms

$$|\mathbf{u}|^p_{W_0^{1,p}(\Omega)^n} = \int_\Omega \sum_{i=1..3} |\partial_i \mathbf{u}|^p,$$

$$|\mathbf{u}|^p_{W_0^{2,p}(\Omega)^n} = \int_\Omega \sum_{i=1..3} |\partial_i \mathbf{u}|^p + \sum_{i,j=1..3} |\partial_i \partial_j \mathbf{u}|^p,$$

where $|\cdot|$ is the Euclidean norm of a vector and ∂_i, $i = 1..3$, is the i^{th} derivative. We will only need the case $\Omega = B(0,1)$. Denote by

$$[\mathbf{u}]_\alpha = \sup_{x,y\in\Omega,\, x\neq y} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{|x-y|^\alpha},\ \alpha \in]0,1[,$$

$$[\mathbf{u}]_{\alpha,\partial\Omega} = \sup_{x,y\in\partial\Omega,\, x\neq y} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{|x-y|^\alpha},\ \alpha \in]0,1[,$$

$$\langle \mathbf{u} \rangle_{\alpha,\partial\Omega} = \sup_{x\in\partial\Omega, y\in\Omega,\, x\neq y} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{|x-y|^\alpha},\ \alpha \in]0,1[,$$

various Hölder seminorms. We will also use the following norm.

$$|\mathbf{u}|_{\alpha,\partial\Omega} = |\mathbf{u}|_\infty + [\mathbf{u}]_{\alpha,\partial\Omega}.$$

The following embedding results hold, see Theorem 7.10, Theorem 7.17 of [2].

Lemma 1. *Let $u \in W_0^{1,2}(\Omega)$. Then, there is $c > 0$, independent of u, such that*

$$|u|_6 \le c|u|_{W_0^{1,2}}.$$

Lemma 2. *Let $\mathbf{u} \in W_0^{1,p}(\Omega)$, $p > 3$. Then, there is $c = c(p)$ such that for a.e. $x, y \in \Omega$ it holds*

$$|u(x) - u(y)| \le c|x-y|^\alpha |\nabla u|_p, \ \alpha = 1 - 3p^{-1}.$$

2.2 Lamé equations of the linear elasticity and Lichtenstein boundary integral equation

In this section we use the classical approach for reduction of the boundary problem for an elliptic system of linear elasticity to an integral equation on the boundary of the domain for the divergence of the displacement field (we use viscous flux, instead), see Ch. 6, [8].

The principal part of (2) is an elliptic system of Lamé equations (19). Consider the problem

$$-(\lambda+\mu)\nabla \mathrm{div}\,\mathbf{u} - \mu\Delta\mathbf{u} = \mathbf{F}, \qquad \Omega, \tag{19}$$

$$u = 0, \qquad \partial\Omega, \tag{20}$$

with the condition (4). The system is $W_0^{1,2}$ - elliptic, see [10](Chap. 3, sec. 7), meaning that the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega (\lambda+\mu)\mathrm{div}\,\mathbf{u}\mathrm{div}\,\mathbf{v} + \mu\nabla\mathbf{u} : \nabla\mathbf{v}, \ \ \mathbf{u},\mathbf{v} \in W_0^{1,2}(\Omega)^3,$$

is coercive, i.e.

$$a(\mathbf{u},\mathbf{u}) \ge \mu|\mathbf{u}|^2_{W_0^{1,2}(\Omega)^3}.$$

This condition is sufficient to imply the existence of the strong solution, see the proof of Theorem 2.1 in [10], Chap. 3.

Theorem 1. *Let $\mathbf{F} \in L^2(\Omega)^3$. Then, there is the unique strong solution of* (19),(20), $\mathbf{u} \in W_0^{2,2}(\Omega)^3$, *such that*

$$|\mathbf{u}|_{W_0^{2,2}(\Omega)^3} \le c|\mathbf{F}|_{L^2(\Omega)^3}.$$

It will be convenient to have an integral representation for the solution (classical) of (19), (20). This can be done in the following way, see Ch. 6, [8]. Let

$$G(x,y) = G^1(x,y) + G^2(x,y) = \begin{cases} \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|y||x-\bar{y}|}, & y \neq 0, \\ \frac{1}{4\pi|x|} + \frac{1}{4\pi}, & y = 0, \end{cases} \tag{21}$$

with $\bar{y} = \frac{y}{|y|^2}$, $y \neq 0$, denote the Green function of the unit ball. Let

$$F = (\lambda + 2\mu)\mathrm{div}\, \mathbf{u} - P + \overline{P}, \tag{22}$$

where $\overline{P} = A\bar{\rho}$. Applying div to (19) we derive:

$$-\Delta F = \mathrm{div}\, \mathbf{F}_1, \tag{23}$$

where $\mathbf{F}_1 = (\rho\mathbf{u})_t - \mathrm{div}\, \rho\mathbf{u}\otimes\mathbf{u}$ and the following integral representation holds (we suppress the dependence of functions on t).

$$F(x) = \int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,\cdot)F(\cdot) + \int_{\Omega} G(x,\cdot)\mathrm{div}\, \mathbf{F}_1(\cdot). \tag{24}$$

Using (23), equations (19) can be written in the following form.

$$-\Delta\left[\frac{\lambda+\mu}{2(\lambda+2\mu)}Fx + \mu\mathbf{u}\right] = \mathbf{F}_1 + \frac{\lambda+\mu}{2(\lambda+2\mu)}\mathrm{div}\, \mathbf{F}_1 x - \frac{\mu}{\lambda+2\mu}\nabla(P-\overline{P}) \tag{25}$$

and so,

$$\mu\mathbf{u}(x) + \frac{\lambda+\mu}{2(\lambda+2\mu)}F(x)x = \frac{\lambda+\mu}{2(\lambda+2\mu)}\int_{\partial\Omega} \partial_{n_y}G(x,y)F(y)y + \int_{\Omega} G(x,y)\left[\mathbf{F}_1 + \frac{\lambda+\mu}{2(\lambda+2\mu)}\mathrm{div}\, \mathbf{F}_1 y - \frac{\mu}{\lambda+2\mu}\nabla(P-\overline{P})\right]. \tag{26}$$

We take div of the last equations and use integral representation (24) for F to get the next equation.

$$\begin{aligned} \frac{3\lambda+5\mu}{2(\lambda+2\mu)}F(x) &+ \frac{\mu}{\lambda+2\mu}(P(x)-\overline{P}) \\ &= \frac{\lambda+\mu}{2(\lambda+2\mu)}\int_{\partial\Omega} \partial_{n_y}\partial_{x_i}G(x,y)F(y)(y_i-x_i) \\ &+ \frac{\lambda+\mu}{2(\lambda+2\mu)}\int_{\Omega} \partial_{x_i}G(x,y)(y_i-x_i)\mathrm{div}\, \mathbf{F}_1(y) \\ &+ \int_{\Omega} \partial_{x_i}G(x,y)\mathbf{F}_1^i(y) - \frac{\mu}{\lambda+2\mu}\int_{\Omega} \partial_{x_i}G(x,y)\partial_{y_i}(P(y)-\overline{P}). \end{aligned} \tag{27}$$

Consider the integral

$$\int_\Omega \partial_{x_i} G(x,y)\partial_{y_i}(P(y)-\overline{P}) = \int_\Omega \partial_{x_i} G(x,y)\partial_{y_i}(P(y)-P(x))$$
$$= \int_\Omega \partial_{x_i} \frac{1}{4\pi|x-y|}\partial_{y_i}(P(y)-P(x))$$
$$+ \int_\Omega \partial_{x_i}\left(-\frac{1}{4\pi|y||x-\bar{y}|}\right)\partial_{y_i}(P(y)-P(x))$$
$$= \int_\Omega \partial_{y_i}\partial_{x_i}\frac{1}{4\pi|y||x-\bar{y}|}(P(y)-P(x)), \quad (28)$$

where we used the fact that $\partial_{x_i} G(x,y) = 0$, $y \in \partial\Omega$, and $-\Delta(4\pi|x-y|)^{-1} = \delta(x-y)$. We obtain from (27) the following equation.

$$\frac{3\lambda+5\mu}{2(\lambda+2\mu)}F(x) + \frac{\mu}{\lambda+2\mu}(P(x)-\overline{P})$$
$$= \frac{\lambda+\mu}{2(\lambda+2\mu)}\int_{\partial\Omega} \partial n_y \partial x_i G(x,y)F(y)(y_i-x_i)\, dS$$
$$+ \frac{\lambda+\mu}{2(\lambda+2\mu)}\int_\Omega \partial_{x_i} G(x,y)(y_i-x_i)\mathrm{div}\,\mathbf{F}_1(y)$$
$$+ \int_\Omega \partial_{x_i} G(x,y)\mathbf{F}_1^i(y) - \frac{\mu}{\lambda+2\mu}\int_\Omega \partial_{y_i}\partial_{x_i} G^2(x,y)(P(y)-P(x)). \quad (29)$$

We would like now to take the limit in the above equation when $x \to y_0 \in \partial\Omega$. A direct computation shows that if $y \in \partial\Omega$, then

$$\frac{\partial^2}{\partial x_i \partial n_y} G(x,y)(x_i-y_i) = \frac{1}{4\pi|x-y|} + \frac{1-|x|^2}{2\pi|x-y|^3},$$

and we deduce that

$$\lim_{x\to y_0}\int_{\partial\Omega} \frac{\partial^2}{\partial n_y \partial x_i} G(x,y)F(y)(y_i-x_i)\, dS = -\frac{1}{4\pi}\int_{\partial\Omega}\frac{F(y)}{|y-y_0|}\, dS - 2F(y_0),$$

and consequently,

$$
\begin{aligned}
\frac{5\lambda+7\mu}{2(\lambda+2\mu)}F(y_0) = & -\frac{\lambda+\mu}{8\pi(\lambda+2\mu)}\int_{\partial\Omega}\frac{F(y)}{|y-y_0|}\,dS \\
& + \frac{\lambda+\mu}{2(\lambda+2\mu)}\int_{\Omega}\partial_{x_i}G(x,y)\big|_{x=y_0}(y_i-y_{0,i})\operatorname{div}\mathbf{F}_1(y) \\
& + \int_{\Omega}\partial_{x_i}G(x,y)\big|_{x=y_0}\mathbf{F}_1^i(y) - \frac{\mu}{(\lambda+2\mu)}(P(y_0)-\overline{P}) \\
& \quad - \frac{\mu}{\lambda+2\mu}\int_{\Omega}\partial_{y_i}\partial_{x_i}G^2(x,y)\big|_{x=y_0}(P(y)-P(y_0)). \quad (30)
\end{aligned}
$$

The last integral converges absolutely since P is C^α "at the boundary" and all singularities, that are of order -3 , of the kernel are restricted to the boundary of Ω. We introduce notation for the singular integral appearing above and some constants.

$$
\begin{aligned}
& K[F](y_0) = \int_{\partial\Omega}\frac{F(y)}{4\pi|y-y_0|}\,dS, \\
& \alpha_1 = \frac{\lambda+\mu}{(5\lambda+7\mu)},\ \alpha_2 = 4\pi\alpha_1,\ \alpha_3 = \frac{2(\lambda+2\mu)}{5\lambda+7\mu},\ \alpha_4 = \frac{2\mu}{5\lambda+7\mu}. \quad (31)
\end{aligned}
$$

We write equations (30) in the following form.

$$
\begin{aligned}
F(y_0)+\alpha_1K[F](y_0) = & \ \alpha_2\int_{\Omega}\partial_{x_i}G(y_0,y)(y_i-y_{0,i})\operatorname{div}\mathbf{F}_1(y) \\
& + \alpha_3\int_{\Omega}\partial_{x_i}G(y_0,y)\mathbf{F}_1^i(y) - \alpha_4(P-\overline{P}) \\
& \quad - \alpha_4\int_{\Omega}\partial_{y_i}\partial_{x_i}G^2(x,y)(P(y)-P(y_0)). \quad (32)
\end{aligned}
$$

The singularity of the kernel of K is integrable and thus K is a smoothing operator. The following lemma is easily verified.

Lemma 3. *K is a bounded linear operator that maps $L^\infty(\partial\Omega)$ to $C^\alpha(\partial\Omega)$, and $C^\alpha(\partial\Omega)$ to $C^{1+\alpha}(\partial\Omega)$, $\alpha\in]0,1[$. Also, $\|K\|_{L^\infty\to L^\infty}\leq 1$.*

It follows then that for any $|\alpha_1|<1$ the resolvent $(I+\alpha_1K)^{-1}$ is a bounded linear operator that maps $L^\infty(\partial\Omega)$ to $L^\infty(\partial\Omega)$ and

$$
|(I+\alpha_1K)^{-1}[f]|_\infty \leq (1-\alpha_1)^{-1}|f|_\infty. \quad (33)
$$

Next, we estimate the norm of K as a map $C^\alpha\to C^\alpha$.

Lemma 4. *Let $f(x)$ in $C^\alpha(\partial\Omega)$, $0<\alpha<1$. Then, $K[f]\in C^\alpha(\partial\Omega)$ and*

$$|K[f]|_{\alpha,\partial\Omega} \le \frac{2+2\alpha}{\alpha}|f|_{\alpha,\partial\Omega}.$$

Proof. . Let $x_1 \neq x_2 \in \partial\Omega$ and $\partial\Omega_1 = \partial\Omega \cap \{|y-x_2| < |y-x_1|\}$, $\partial\Omega_2 = \partial\Omega \setminus \partial\Omega_1$. We have,

$$\begin{aligned} K[f](x_2)-K[f](x_1) &= \int_{\partial\Omega_1} (f(y)-f(x_2))\left\{\frac{1}{4\pi|y-x_2|}-\frac{1}{4\pi|y-x_1|}\right\} dS \\ &+ \int_{\partial\Omega_2} (f(y)-f(x_1))\left\{\frac{1}{4\pi|y-x_2|}-\frac{1}{4\pi|y-x_1|}\right\} dS \\ &+ f(x_2)\int_{\partial\Omega_1} \left\{\frac{1}{4\pi|y-x_2|}-\frac{1}{4\pi|y-x_1|}\right\} dS \\ & f(x_1)\int_{\partial\Omega_2} \left\{\frac{1}{4\pi|y-x_2|}-\frac{1}{4\pi|y-x_1|}\right\} dS. \end{aligned} \quad (34)$$

Note, that

$$\begin{aligned} &\int_{\partial\Omega_1} \left\{\frac{1}{4\pi|y-x_2|}-\frac{1}{4\pi|y-x_1|}\right\} dS \\ &\quad = -\int_{\partial\Omega_2} \left\{\frac{1}{4\pi|y-x_2|}-\frac{1}{4\pi|y-x_1|}\right\} dS \le 2, \end{aligned} \quad (35)$$

where in the last inequality we used the identity $\int_{\partial\Omega} \frac{1}{4\pi|y-x_1|}\, dS = 1$, $\forall x_1 \in \partial\Omega$. Also, since

$$|y-x_2|^{-1} - |y-x_2|^{-1} \le \frac{1}{\alpha z^{1+\alpha}}|x_1-x_2|^\alpha, \quad (36)$$

for some $0 < z \in [\min\{|y-x_2|, |y-x_1|\}, \max\{|y-x_2|,|y-x_1|\}]$. It follows from (34) and (35) that

$$K[f](x_2) - K[f](x_1) \le \frac{2(1+\alpha)}{\alpha}[f]_{\alpha,\partial\Omega}|x_2-x_1|^\alpha.$$

But, it is also obvious that $\sup|K[f](x)| \le \sup|f(x)|$. The lemma is proven. □

Operator K considered on C^α, $\alpha>0$, is a compact operator and thus has only discrete spectrum. Moreover, K is a smoothing operator as stated in Lemma 3 and any its eigenfunction is an element of C^∞.

By Lemma 4 it follows that $\alpha_1 K$ is a contraction map when, in particular, $\alpha_1 \in [0, 1/5]$ and $\alpha > 2/3$. It follows that $\alpha_1 \in [0, 1/5]$ is in the resolvent set for all $\alpha > 0$. The value of α_1 is determined by $3\lambda + 2\mu \geq 0$, $\mu > 0$ and is given in (31). We see that $\alpha_1 \in [1/7, 1/5[$, that is, belongs to the resolvent set of K. Thus, the resolvent, $(I + \alpha_1 K)^{-1}$ is uniformly bounded for $\alpha_1 \in [1/7, 1/5]$, i.e. there is $c^1 = c^1(\alpha)$ such that

$$|| \, (I + \alpha_1 K)^{-1} \, ||_{C^\alpha \to C^\alpha} \leq c^1(\alpha). \tag{37}$$

Now, we solve equation (32) for F and using estimates (33), (37) and results of lemma 7 and 8, we have the following representation for the boundary values of normal stress F.

$$\begin{aligned} F(t, y) &= \partial_t \, (I + \alpha_1 K)^{-1} \, [\alpha_2 J_1 + \alpha_3 K_1](t, y) \\ &\quad + \sum_2^4 \, (I + \alpha_1 K)^{-1} \, [\alpha_2 J_j + \alpha_3 K_j](t, y) \\ &- \alpha_4 \, (I + \alpha_1 K)^{-1} \, [P - \overline{P}](t, y) - \alpha_4 \, (I + \alpha_1 K)^{-1} \, [P_1](t, y) \\ &\triangleq \partial_t g(t, y) + f(t, y), \end{aligned} \tag{38}$$

and

$$\begin{aligned} [g(\tau, \cdot)]_{\gamma, \partial\Omega} &\leq c(p) |\rho(\tau, \cdot)|_\infty |\mathbf{u}(t, \cdot)|_p, & (39) \\ |f(\tau, \cdot)|_{\infty, \partial\Omega} &\leq c |\rho(\tau, \cdot)|_\infty |\mathbf{u}(\tau, \cdot)|_\infty [\mathbf{u}(\tau, \cdot)]_{1/2} \\ &+ \delta \langle P(\tau, \cdot) \rangle_{\alpha, \partial\Omega} + c_\delta |P(\tau, \cdot) - \overline{P}|_\infty, & (40) \\ [f(\tau, \cdot)]_{\alpha, \partial\Omega} &\leq c(\alpha) |\rho(\tau, \cdot)|_\infty |\mathbf{u}(\tau, \cdot)|_\infty [\mathbf{u}(\tau, \cdot)]_{1/2} \\ &+ c(\alpha) \alpha_4 \langle P(\tau, \cdot) \rangle_{\alpha, \partial\Omega}, \\ &\gamma = 1 - 3p^{-1}, \, p > 3, \quad \alpha \in]0, 1/2[, & (41) \end{aligned}$$

where c on the right side does not depend on λ, μ. Note, that we keep α_4 in the estimates in order to compare corresponding terms with the damping effect of pressure in the equation of evolution of density. Now, we estimate F at the interior point $x \in \Omega$, where the values of F at the boundary are determined by (38). For this, consider first the following integral.

$$\int_{\partial\Omega} \partial_{n_y} G(x,\cdot)\partial_t \left(I+\alpha_1 K\right)^{-1}[J_1+K_1](t,\cdot)\,dS =$$
$$\frac{d}{dt}\int_{\partial\Omega} \partial_{n_y} G(x,\cdot)\left(I+\alpha_1 K\right)^{-1}[J_1+K_1](t,\cdot)\,dS$$
$$+\int_{\partial\Omega} \partial_{x_i}\partial_{n_y} G(x,\cdot)\left(I+\alpha_1 K\right)^{-1}[J_1+K_1](t,\cdot)\left\{u_i(t,\cdot)-u_i(t,x)\right\}. \tag{42}$$

Here we used the fact that $\mathbf{u}=0$ at the boundary of Ω. Using this and (38) in (24) we write the next representation.

$$F(t,x) = \frac{d}{dt}\int_{\partial\Omega} \partial_{n_y} G(x,\cdot) g(t,\cdot)\,dS$$
$$+\int_{\partial\Omega} \partial_{x_i}\partial_{n_y} G(x,\cdot) g(t,\cdot)\left\{u_i(t,\cdot)-u_i(t,x)\right\}\,dS$$
$$+\int_{\partial\Omega} \partial_{n_y} G(x,\cdot) f(t,\cdot)\,dS - \frac{d}{dt}K_1(x)+\sum_2^4 K_2(x) \triangleq \frac{d}{dt}h(t,x)+l(t,x), \tag{43}$$

where we used the relation $-\int_\Omega G(x,\cdot)\mathrm{div}\,\mathbf{F}_1(\cdot) = \frac{d}{dt}K_1(x)+\sum_2^4 K_2(x)$, where all K_i are estimated by lemma (8). The use of lemma 9 and 10 with estimates (39) and (41) results in the following estimates.

$$\begin{aligned}
[h(\tau,\cdot)]_\gamma &\leq c(p)|\rho(\tau,\cdot)|_\infty |\mathbf{u}(t,\cdot)|_p \qquad (44)\\
[l(\tau,\cdot)]_\alpha &\leq c(\alpha)|\rho(\tau,\cdot)|_\infty |[\mathbf{u}(\tau,\cdot)]_{1/2}^2\\
&+ c(\alpha)\alpha_4\langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega},\\
&\gamma = 1-3p^{-1},\ p>3, \quad \alpha\in]0,<1/2[,
\end{aligned}$$

where c on the right side does not depend on λ, μ. Moreover, using lemma 1 and lemma 5 we obtain the next estimate for l.

$$\begin{aligned}
[l(\tau,\cdot)]_\alpha &\leq c|\rho(\tau,\cdot)|_\infty |(\langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega}^2 + |\mathbf{F}_1(\tau,\cdot)|_2^2)\\
&+ c_1(\alpha)\alpha_4\langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega},\\
&\alpha\in]0,1/2[, \qquad (45)
\end{aligned}$$

where c_1 still does not depend on either λ or μ.

We will also need the L^p estimates for $\nabla\mathbf{u}$.

Lemma 5. *Let* $\mathbf{u}$ *be the solution of* (19)-(20). *Then, there is* $c > 0$, *such that*

$$|\nabla \mathbf{u}|_6 \leq c\langle P\rangle_{\alpha,\partial\Omega} + c|\mathbf{F}_1|_2. \tag{46}$$

Proof. We write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ satisfies the Lamé equations (19) with $\nabla(P - \overline{P})$ as a forcing term and $\mathbf{u}_2$ satisfies the Lamé equations (19) with F_1 as a forcing term; both have zero boundary values. Then, by the Theorem 2.1 in [10], Ch. 3, it holds that $|\mathbf{u}_2|_{W_0^{2,2}} \leq c|\mathbf{F}_1|_2$ and the estimate for $\nabla \mathbf{u}_2$ follows from Poincaré-Sobolev inequality. The estimate for $\nabla \mathbf{u}_1$ follows the representation of $\mathbf{u}_1$ which is similar to (23), (26), i.e. if $\bar{F} = (\lambda + 2\mu)\mathrm{div}\, \mathbf{u}_1 - (P - \overline{P})$ then

$$\begin{aligned}
\Delta \bar{F} &= 0,\\
\mu \mathbf{u}_1(x) + \frac{\lambda+\mu}{2(\lambda+2\mu)}\bar{F}(x)x &= \frac{\lambda+\mu}{2(\lambda+2\mu)}\int_{\partial\Omega} \partial n_y G(x,y)\bar{F}(y)y\, dS\\
&- \frac{\mu}{\lambda+2\mu}\int_{\Omega} G(x,y)\nabla(P(y) - \overline{P}),
\end{aligned}$$

and the line of argument that resulted in (45) and classical Calderón-Zygmund estimate on singular integrals for $\nabla_x \int_\Omega G(x,y)\nabla(P(y) - \overline{P})$ (the details are omitted). □

2.3 Lipschitz continuity of flow trajectories near the boundary

In this section we derive the estimate on $|\nabla \mathbf{u}|_\infty$ near the boundary.

Let $X(t,\tau;x)$, $X(t,t;x) = x$, $x \in \Omega$ denote the trajectory of flow generated by $\mathbf{u}(t,x)$, i.e.

$$\frac{d}{d\tau}X(t,\tau;x) = \mathbf{u}(\tau, X(t,\tau;x)).$$

We abbreviate $X_{-t}(x)$ for $X(t,0;x)$. For $x_1 \in \partial\Omega$, $x_2 \in \Omega$ we set $d(t-\tau) = |X(t,\tau;x_1) - X(t,\tau;x_2)|$. Using (24) we write (26) as

$$\begin{aligned}
\mu\mathbf{u}(x) = \beta_1 &\int_{\partial\Omega} \partial_{n_y} G(x,\cdot)(\cdot - x)F(\cdot)\, dS\\
&+ \beta_1 \int_\Omega G(x,\cdot)(\cdot - x)\mathrm{div}\, \mathbf{F}_1(\cdot) + \int_\Omega G(x,\cdot)\mathbf{F}_1(\cdot)\\
&\quad - \beta_2 \int_\Omega G(x,\cdot)\nabla(P(\cdot) - \overline{P}) \triangleq \mu\mathbf{u}^1 + \mu\mathbf{u}^2 + \mu\mathbf{u}^3, \quad (47)
\end{aligned}$$

where $\beta_1 = \frac{\lambda+\mu}{2(\lambda+2\mu)}$, $\beta_2 = \frac{\mu}{\lambda+2\mu}$. The boundary values of $F(y)$ are split into the sum $\partial_t g(t,y) + f(t,y)$ according to formula (38). Then, we can write

$$\frac{\mu}{\beta_1}\mathbf{u}^1 = \frac{d}{dt}\int_{\partial\Omega} \partial_{n_y} G(x,\cdot)(\cdot - x) g(t,\cdot)\, dS + \int_{\partial\Omega} \partial_{x_i}\left[\partial_{n_y} G(x,\cdot)(\cdot - x)\right] g(t,\cdot)\{u_i(t,\cdot) - u_i(t,x)\}\, dS + \int_{\partial\Omega} \partial_{n_y} G(x,\cdot)(\cdot - x) f(t,\cdot)\, dS. \quad (48)$$

We assume from this point on that the following hypothesis holds.

$$\text{Hypothesis } H_1 : \quad \check{\rho} < \rho(t,x) < \hat{\rho}, \quad (49)$$

for some $\check{\rho}$, $\hat{\rho} > 0$ and $t \geq 0, x \in \Omega$. Estimate (37) together with the second parts of lemma 84 and 9 yields the following estimate.

$$\int_0^t \mathbf{u}^1(\tau, X(t,\tau;x_1)) - \mathbf{u}^1(\tau, X(t,\tau;x_2)) \leq c\left([g(t,\cdot)]_\alpha d(0) + [g(0,\cdot)]_\alpha d(t)\right) + c\int_0^t [g(\tau,\cdot)]_\omega d(t-\tau) + \varepsilon \int_0^t [f(\tau,\cdot)]_{\alpha,\partial\Omega} d(t-\tau) + c_\varepsilon \int_0^t |f(\tau,\cdot)|_{\infty,\partial\Omega} d(t-\tau),$$

where $0 < \alpha < 1$, $1/2 < \omega < 1$, and $c = c(\alpha, \omega, \lambda, \mu)$. Then, restricting $\alpha < 1/2$ and $\omega = 1 - 3/p_1$, $p_1 > 6$ and $\gamma = 1 - 3/p$, $p > 3$, from (39), (40) and (41) we conclude

$$\begin{aligned}\int_0^t \mathbf{u}^1(\tau, X(t,\tau;x_1)) - \mathbf{u}^1(\tau, X(t,\tau;x_2)) &\leq c\left(|\rho(t,\cdot)|_\infty |\mathbf{u}(t,\cdot)|_p d(0) + |\rho_0|_\infty |\mathbf{u}_0|_p d(t)\right) \\ &+ \int_0^t |\rho(\tau,\cdot)|_\infty |\mathbf{u}(\tau,\cdot)|_{p_1} d(t-\tau) \\ &+ c\int_0^t |\rho(\tau,\cdot)|_\infty [\mathbf{u}(\tau,\cdot)]_{1/2}^2 d(t-\tau) \\ &+ (\varepsilon + \delta)\int_0^t \langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega} d(t-\tau) \\ &+ (c_\varepsilon + c_\delta)\int_0^t |P(\tau,\cdot) - \overline{P}|_\infty d(t-\tau), \quad (50)\end{aligned}$$

for some $c = c(\alpha, p, p_1, \lambda, \mu)$.

We use (86) to estimate $\mathbf{u}^3$.

$$\int_0^t \mathbf{u}^3(\tau, X(t,\tau;x_1)) - \mathbf{u}^3(\tau, X(t,\tau;x_2))$$
$$\leq \varepsilon \int_0^t \langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega} d(t-\tau) + c_\varepsilon \int_0^t |P(\tau,\cdot) - \overline{P}|_\infty d(t-\tau). \quad (51)$$

Sice, $\mathbf{F}_1 = (\rho\mathbf{u})_t + \operatorname{div}\rho\mathbf{u}\otimes\mathbf{u}$, we write $\mathbf{u}^2$ in the following form (we suppress the dependence on t temporally).

$$\begin{aligned}
\mu\beta_1^{-1}\mathbf{u}_k^2(x) = &-\frac{d}{dt}\int \partial_{y_j}\{G(x,y)(x_k-y_k)\}\rho(y)u_j(y) \\
&+ u_m(x)\partial_{x_m}\int \partial_{y_j}\{G(x,y)(x_k-y_k)\}\rho(y)u_j(y) \\
&+ \int \partial_{y_m}\partial_{y_j}\{G(x,y)(x_k-y_k)\}\rho(y)u_j(y)u_m(y) \\
&-\frac{d}{dt}\int G(x,y)\rho(y)u_k(y) + u_m(x)\partial_{x_m}\int G(x,y)\rho(y)u_k(y) \\
&+ \int \partial_{y_m}G(x,y)\rho(y)u_k(y)u_m(y) \\
&= \frac{d}{dt}(\bar{J}_1 + \bar{K}_1) + \sum_2^4 \bar{J}_i + \bar{K}_i, \quad (52)
\end{aligned}$$

where (we suppress the dependence on $k = 1..3$)

$$\begin{aligned}
\bar{J}_1(x) &= -\int \partial_{y_j}\{G(x,y)(x_k-y_k)\}\rho(y)u_j(y), \\
\bar{J}_2(x) &= \int \partial_{y_m}\partial_{y_j}\left\{G^1(x,y)(x_k-y_k)\right\}\rho(y)u_j(y)[u_m(y)-u_m(x)], \\
\bar{J}_3(x) &= u_m(x)\partial_{x_m}\int \partial_{y_j}\left\{G^2(x,y)(x_k-y_k)\right\}\rho(y)u_j(y), \\
\bar{J}_4(x) &= \int \partial_{y_m}\partial_{y_j}\left\{G^2(x,y)(x_k-y_k)\right\}\rho(y)u_j(y)u_m(y)
\end{aligned}$$

and

$$\begin{aligned}
\bar{K}_1(x) &= -\int G(x,y)\rho(y)u_k(y), \\
\bar{K}_2(x) &= \int \partial_{y_m}\partial_{y_j} G^1(x,y)\rho(y)u_j(y)[u_m(y)-u_m(x)], \\
\bar{K}_3(x) &= u_m(x)\partial_{x_m}\int \partial_{y_j} G^2(x,y)\rho(y)u_j(y), \\
\bar{K}_4(x) &= \int \partial_{y_m}\partial_{y_j} G^2(x,y)\rho(y)u_j(y)u_m(y).
\end{aligned}$$

Similarly to the proofs of lemma 7 and lemma 8 we obtain that for any $t>0$,

$$|\nabla \bar{J}_1(t,\cdot)|_\infty + |\nabla \bar{K}_1|_\infty \le c|\rho(t,\cdot)\mathbf{u}(t,\cdot)|_6 \le c|\nabla \mathbf{u}(t,\cdot)|_2,$$

where $c=c(\alpha,\lambda,\mu,\hat{\rho})$ and the last inequality is an instance of Poincaré-Sobolev inequality. Also,

$$|\nabla \bar{J}_i(t,\cdot)|_\infty(|\bar{K}_1|_\infty) \le c|\rho(t,\cdot)\mathbf{u}(t,\cdot)|_\infty[\mathbf{u}]_{\frac{1}{2}} \le c[\mathbf{u}(t,\cdot)]^2_{\frac{1}{2}}.$$

The last two estimates are used to get

$$\begin{aligned}
\int_{t-\tau}^{t} \mathbf{u}^2(\tau,X(t,z;x_1)) - \mathbf{u}^2(\tau,X(t,z;x_2)) &\le c|\nabla\mathbf{u}(t,\cdot)|_2 d(0) \\
&+ c|\nabla\mathbf{u}(t-\tau,\cdot)|_2 d(\tau) + c\int_{t-\tau}^{t}[\mathbf{u}(z,\cdot)]^2_{\frac{1}{2}} d(t-z). \quad (53)
\end{aligned}$$

Combining (50), (51) and (53) we can derive the following integral inequality for $d(\cdot)$.

$$\begin{aligned}
d(\tau) \le d(0) &+ c|\nabla\mathbf{u}(t,\cdot)|_2 d(0) \\
&+ c|\nabla\mathbf{u}(t-\tau,\cdot)|_2 d(\tau) + c\int_{t-\tau}^{t}|\nabla\mathbf{u}(z,\cdot)|_2 d(t-z) \\
&+ c\int_{t-\tau}^{t}[\mathbf{u}(z,\cdot)]^2_{\frac{1}{2}} d(t-z) + (\varepsilon+\delta)\int_{t-\tau}^{t}\langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega} d(t-z) \\
&+ c(\varepsilon,\delta)\int_{t-\tau}^{t}|P(\tau,\cdot)-\overline{P}|_\infty d(t-z).
\end{aligned}$$

From here we deduce that there is $c=c(\alpha,\lambda,\mu,\hat{\rho})$ such that

$$d(\tau) \le A_0|x_1-x_2|e^{c\int_{t-\tau}^{t} X(\cdot)}, \quad \tau\in[0,t], \tag{54}$$

where

$$A_0 = \frac{1 + \sup_{[0,\infty[} |\nabla \mathbf{u}|_2}{1 - \sup_{[0,\infty[} |\nabla \mathbf{u}|_2},$$

$$X(\tau) = |\nabla \mathbf{u}(\tau, \cdot)|_2 + [\mathbf{u}(\tau, \cdot)]^2_{\frac{1}{2}} + (\varepsilon + \delta)\langle P(\tau, \cdot)\rangle_{\alpha,\partial\Omega} + c(\varepsilon, \delta)|P(\tau, \cdot) - \overline{P}|_\infty.$$

2.4 Some potential estimates

This section constitutes somewhat technical part of the paper, devoted to the derivation of some potential type estimates for the singular integrals appearing in equation (32), where we take $\mathbf{F}_1 = (\rho\mathbf{u})_t + \operatorname{div} \rho\mathbf{u} \otimes \mathbf{u}$.

It is a straightforward computation to verify that $\partial_{x_i}\partial_{n_y} G^2(x, y)(x_i - y_i)$ can be written as $k(x, y)(|y||x - \bar{y}|)^{-3}$, where $k(x, y)$ is uniformly bounded. Let

$$P_1(x) = \int_\Omega \frac{k(x, y)}{|y|^3|x - \bar{y}|^3} \{P(y) - P(x)\}.$$

Lemma 6. *There is $c = c(\alpha)$ such that*

$$\langle P_1\rangle_{\alpha,\partial\Omega} \leq c\langle P\rangle_{\alpha,\partial\Omega} \tag{55}$$

and $\forall\delta > 0$ there is $c_\delta = c_\delta(\alpha)$ such that

$$|P_1|_\infty \leq \delta\langle P\rangle_{\alpha,\partial\Omega} + c_\delta|P - \overline{P}|_\infty. \tag{56}$$

Proof. We proof only the first part of the lemma. The proof of the second part is trivial. Let $\xi_1 \in \partial\Omega$, $\xi_2 \in \Omega$ and $|\xi_1 - \xi_2| < \frac{1}{8}$. Let $\xi_2' = \xi_2/|\xi_2|$ and $\xi_0 \in \partial\Omega$ is such that $|\xi_0 - \xi_1| = |\xi_0 - \xi_2'|$. Denote by

$B = B(\xi_0,\, 2|\xi_1 - \xi_2|)$ and $B_1 = B(\xi_0,\, \frac{1}{2})$. We can write

$$\begin{aligned} P_1(\xi_1) - P_1(\xi_2) = &\int_{\Omega\cap B} \frac{k(\xi_1,y)}{|y|^3|\xi_1-\bar{y}|^3}\{P(y)-P(\xi_1)\} \\ &- \int_{\Omega\cap B} \frac{k(\xi_2,y)}{|y|^3|\xi_2-\bar{y}|^3}\{P(y)-P(\xi_2)\} \\ + \int_{\Omega\setminus B_1} &\frac{k(\xi_1,y)}{|y|^3|\xi_1-\bar{y}|^3}\{P(y)-P(\xi_1)\} - \int_{\Omega\setminus B_1} \frac{k(\xi_2,y)}{|y|^3|\xi_2-\bar{y}|^3}\{P(y)-P(\xi_2)\} \\ &+ \{P(\xi_0)-P(\xi_1)\}\int_{\Omega\cap B_1\setminus B} \frac{k(\xi_1,y)}{|y|^3|\xi_1-\bar{y}|^3} \\ &- \{P(\xi_0)-P(\xi_2)\}\int_{\Omega\cap B_1\setminus B} \frac{k(\xi_2,y)}{|y|^3|\xi_2-\bar{y}|^3} \\ &+ \int_{\Omega\cap B_1\setminus B} \left\{\frac{k(\xi_1,y)}{|y|^3|\xi_1-\bar{y}|^3} - \frac{k(\xi_2,y)}{|y|^3|\xi_2-\bar{y}|^3}\right\}\{P(y)-P(\xi_0)\} \\ &= \sum_{i=1}^{7} I_i. \end{aligned} \tag{57}$$

Since $|y||\xi_1 - \bar{y}| = |\xi_1 - y|$, it straightforward to see that

$$|I_1| \leq c\langle P\rangle_{\alpha,\partial\Omega}|\xi_1 - \xi_2|^{\alpha}. \tag{58}$$

For $y \in \Omega \cap B$, there is a numerical constant c such hat $c|y||\xi_2 - \bar{y}| > |y||\xi_2' - \bar{y}| = |\xi_2' - y|$, and also $|\xi_2 - \bar{y}| > |\xi_2' - \xi_2|$. Thus, we can estimate

$$\begin{aligned} I_2 = &\int_{\Omega\cap B} \frac{k(\xi_2,y)}{|y|^3|\xi_2-\bar{y}|^3}\left\{P(y)-P(\xi_2')\right\} \\ &+ \left\{P(\xi_2')-P(\xi_2)\right\}\int_{\Omega\cap B} \frac{k(\xi_2,y)}{|y|^3|\xi_2-\bar{y}|^3} \\ &\leq c\int_{\Omega\cap B} \frac{|P(y)-P(\xi_2')|}{|\xi_2'-y|^3} \\ &+ c|P(\xi_2')-P(\xi_2)|\int_{\Omega\cap B} \frac{1}{|\xi_2'-\xi_2|^{\alpha}|\xi_2'-y|^{3-\alpha}} \\ &\leq c\langle P\rangle_{\alpha,\partial\Omega}|\xi_1-\xi_2|^{\alpha}. \end{aligned} \tag{59}$$

Next,

$$I_3 + I_4 = \{P(\xi_0) - P(\xi_1)\} \int_{\Omega\backslash B_1} \frac{k(\xi_1, y)}{|\xi_1 - \bar{y}|^3}$$
$$- \{P(\xi_0) - P(\xi_2)\} \int_{\Omega\backslash B_1} \frac{k(\xi_2, y)}{|\xi_2 - \bar{y}|^3}$$
$$+ \int_{\Omega\backslash B_1} \left\{ \frac{k(\xi_1, y)}{|\xi_1 - \bar{y}|^3} - \frac{k(\xi_2, y)}{|\xi_2 - \bar{y}|^3} \right\} \{P(y) - P(\xi_0)\}$$
$$\le c\langle P\rangle_{\alpha,\partial\Omega} |\xi_1 - \xi_2|^\alpha. \quad (60)$$

Also,

$$I_6 = \{P(\xi_0) - P(\xi_2)\} \int_{\Omega\cap B_1\backslash B} \partial_{y_i}\partial_{x_i} G^2(\xi_2, y)$$
$$= \{P(\xi_0) - P(\xi_2)\} \int_{\partial(\Omega\cap B_1\backslash B)} \partial_{x_i} G^2(\xi_2, y) n_{y,i}$$
$$= \{P(\xi_0) - P(\xi_2)\} \int_{\partial\Omega\cap(B_1\backslash B)} \partial_{x_i} G^1(\xi_2, y) n_{y,i}$$
$$+ \{P(\xi_0) - P(\xi_2)\} \int_{\partial B_1\cap\Omega} \partial_{x_i} G^2(\xi_2, y) n_{y,i}$$
$$+ \{P(\xi_0) - P(\xi_2)\} \int_{\partial B\cap\Omega)} \partial_{x_i} G^1(\xi_2, y) n_{y,i}$$
$$= \{P(\xi_0) - P(\xi_2)\} \left\{ I_5^1 + I_5^2 + I_5^3 \right\}. \quad (61)$$

When $y \in \partial\Omega$, $8\pi\partial_{x_i} G^1(\xi_2, y) n_{y,i} = |\xi_2 - y|^{-1} + (|\xi_2|^2 - 1)|\xi_2 - y|^3$. Then

$$|8\pi I_5^1| \le \int_{\partial\Omega\backslash B} \frac{1}{|\xi_2 - y|} + (1 - |\xi_2|^2) \int_{\partial\Omega\backslash B} \frac{1}{|\xi_2 - y|^3} \le c.$$

Also, since $|\xi_2 - \bar{y}| > c$, $y \in \partial B_1$, we have

$$|I_5^2| \le c,$$

Lastly,

$$|I_5^3| \le c \int_{\partial B\cap\Omega} \frac{1}{|\xi_2 - y|^2} \le c.$$

Combining the last three estimates in (61) we derive

$$|I_5| \le c\langle P\rangle_{\alpha,\partial\Omega} |\xi_1 - \xi_2|^\alpha. \quad (62)$$

To estimate I_7 we first notice that

$$\frac{k(\xi_1,y)}{|y|^3|\xi_1-\bar{y}|^3}-\frac{k(\xi_2,y)}{|y|^3|\xi_2-\bar{y}|^3}\le c\frac{|\xi_1-\xi_2|}{|y|^4|\tilde{\xi}-\bar{y}|^4},\ y\neq 0, \tag{63}$$

for some $\tilde{\xi}$ from the linear segment between ξ_1 and ξ_2. Moreover, for $y\in\Omega\setminus B$ it holds that $|\xi_0-\bar{y}|>2|\xi_1-\xi_2|>2|\xi_0-\tilde{\xi}|$. Then we get $|\tilde{\xi}-\bar{y}|>|\xi_0-\bar{y}|-|\xi_0-\tilde{\xi}|>|\xi_0-\bar{y}|/2$. Then,

$$\begin{aligned} I_7 &\le c\langle P\rangle_{\alpha,\partial\Omega}|\xi_1-\xi_2|^{1+\alpha}\int_{\Omega\cap B_1\setminus B}\frac{1}{|\xi_0-y|^4} \\ &\le c\langle P\rangle_{\alpha,\partial\Omega}|\xi_1-\xi_2|^{1+\alpha}\int_{2|\xi_1-\xi_2|}^{2} r^{-2}\,dr\le c\langle P\rangle_{\alpha,\partial\Omega}|\xi_1-\xi_2|^{\alpha}. \end{aligned} \tag{64}$$

□

Lets consider terms on left-hand side of (32). We compute

$$\begin{aligned} \partial_{x_i}G(x,y)(y_i-x_i) &= \frac{1}{4\pi|x-y|}-\frac{(x_i-\bar{y}_i)(x_i-y_i)}{4\pi|y||x-\bar{y}|^3} \\ &\triangleq G_2(x,y)\triangleq G_{2,1}(x,y)+G_{2,2}(x,y). \end{aligned} \tag{65}$$

Moreover, by a straightforward computation, we deduce that

$$\partial_{y_i}\partial_{y_j}G_{2,1}=\frac{k^1_{i,j}(x,y)}{|x-y|^3},\ \partial_{y_i}\partial_{y_j}G_{2,2}=\frac{k^2_{i,j}(x,y)}{|y|^3|x-\bar{y}|^3},$$

for $i,j=1..3$, where $k^l_{i,j}(x,y)$ are uniformly bounded in $x,y\in\Omega$. Now, we have (we suppress writing t as argument of functions)

$$\begin{aligned} \int_\Omega G_2(x,y)\mathrm{div}\,\mathbf{F}(y) &= -\frac{d}{dt}\int_\Omega \nabla_y G_2\cdot\rho(y)\mathbf{u}(y) \\ &+u_j(x)\partial_{x_j}\int_\Omega \partial_{y_i}G_{2,1}(x,y)\rho(y)u_i(y)-\int_\Omega \partial_{y_i}G_{2,1}(x,y)\partial_{y_j}\left[\rho(y)u_i(y)u_j(y)\right] \\ &+u_j(x)\partial_{x_j}\int_\Omega \partial_{y_i}G_{2,2}(x,y)\rho(y)u_i(y)-\int_\Omega \partial_{y_i}G_{2,2}(x,y)\partial_{y_j}\rho(y)u_i(y)u_j(y) \\ &= -\frac{d}{dt}\int_\Omega \nabla_y G_2\cdot\rho(y)\mathbf{u}(y)+\int_\Omega \partial_{y_j}\partial_{y_i}G_{2,1}(x,y)\left[\rho(y)u_i(y)(u_j(y)-u_j(x))\right] \\ &+u_j(x)\int_\Omega \partial_{x_j}\partial_{y_i}G_{2,2}(x,y)\rho(y)u_i(y)+\int_\Omega \partial_{y_j}\partial_{y_i}G_{2,2}(x,y)\rho(y)u_i(y)u_j(y) \\ &= \frac{d}{dt}J_1(x)+J_2(x)+J_3(x)+J_4(x), \end{aligned} \tag{66}$$

where we used the formula for $G_{2,1}$ given by (65). We prove the following lemma.

Lemma 7. *There are $c_0 = c_0(\alpha)$, $c = c(\alpha,\beta)$ and $c_1 = c_1(\gamma)$ such that*

$$|J_i|_\infty \le c_0|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha, \tag{67}$$

$$[J_i]_\beta \le c_1|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha, \quad \beta \in]0,\alpha[,\ i = 2,3,4. \tag{68}$$

$$|J_1|_\infty + [J_1]_\gamma \le c_1|\rho|_\infty|\mathbf{u}|_p, \quad \gamma = 1-3p^{-1},\ p > 3. \tag{69}$$

Proof. Few words about the strategy of the proof. First, one notices that J_2 has the form of commutator; the smoothing properties of it played an important role in [4], [6]. We see that the kernel $\partial_{y_i}\partial_{y_j}G_{2,1}$ has a singularity of order -3 at any point $x \in \Omega$, but it is multiplied by $\mathbf{u}(x)-\mathbf{u}(y)$, which is of order $|x-y|^\alpha$ and classical potential estimates apply to this integral. Secondly, the singularity in the integral representation of J_3 and J_4 are restricted to the boundary of Ω, where $\mathbf{u}$ vanishes, thus these terms can be treated similarly to the first case.

Let x_1, $x_2 \in \Omega$ and $|x_1 - x_2| \le 1/4$. Let $x_0 = \frac{x_1+x_2}{2}$ and $B = B(x_0; 2|x_1-x_2|)$. We can write

$$\begin{aligned}
J_2(x_1) - J_2(x_2) &= \int_{\Omega\cap B} \frac{k^1_{i,j}(x_1,y)}{|x_1-y|^3}\left[\rho(y)u_i(y)(u_j(x)-u_j(y))\right] \\
&\quad - \int_{\Omega\cap B} \frac{k^1_{i,j}(x_2,y)}{|x_2-y|^3}\left[\rho(y)u_i(y)(u_j(x)-u_j(y))\right] \\
&\quad + \int_{\Omega\setminus B}\left\{\frac{k^1_{i,j}(x_1,y)}{|x_1-y|^3} - \frac{k^1_{i,j}(x_2,y)}{|x_2-y|^3}\right\}\rho(y)u_i(y)(u_j(x_0)-u_j(y)) \\
&\quad + (u_j(x_1)-u_j(x_0))\int_{\Omega\setminus B}\frac{k^1_{i,j}(x_1,y)}{|x_1-y|^3}\rho(y)u_i(y) \\
&\quad - (u_j(x_2)-u_j(x_0))\int_{\Omega\setminus B}\frac{k^1_{i,j}(x_2,y)}{|x_2-y|^3}\rho(y)u_i(y) \triangleq \sum_1^5 J_2^i.
\end{aligned} \tag{70}$$

It is easy to see that $|J_2^1|$ and $|J_2^2|$ are bounded by $c|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha|x_1 - x_2|^\alpha$, for suitable c. The same is true about $|J_2^3|$ by the way of the estimate

$$\frac{k^1_{i,j}(x_1,y)}{|x_1-y|^3} - \frac{k^1_{i,j}(x_2,y)}{|x_2-y|^3} \le \frac{c}{|x_0-y|^4}|x_1-x_2|,\ y \in \Omega. \tag{71}$$

For J_2^4(and J_2^5) we have the following estimate.

$$J_2^4 \leq c|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha|x_1-x_2|^\alpha \log|x_1-x_2|^{-1}.$$

We proved estimate (68) for J_2. Lets consider J_3 (J_4 is estimated in the same way). Let x_i, $i=0,1,2$ and B be chosen as above. Consider two cases. First, assume that $B=B\cap\Omega$.

$$J_3(x_1)-J_3(x_2)=\int_{\Omega\setminus B}\left\{\frac{k_{i,j}^2(x_1,y)}{|y|^3|x_1-\bar{y}|^3}-\frac{k_{i,j}^2(x_2,y)}{|y|^3|x_2-\bar{y}|^3}\right\}\rho(y)u_i(y)u_j(y)$$
$$+\int_B\left\{\frac{k_{i,j}^2(x_1,y)}{|y|^3|x_1-\bar{y}|^3}-\frac{k_{i,j}^2(x_2,y)}{|y|^3|x_2-\bar{y}|^3}\right\}\rho(y)u_i(y)u_j(y)\triangleq J_3^1+J_3^2. \tag{72}$$

For $y\in\Omega\setminus B$ we have the following estimate.

$$\frac{k_{i,j}^2(x_1,y)}{|y|^3|x_1-\bar{y}|^3}-\frac{k_{i,j}^2(x_2,y)}{|y|^3|x_2-\bar{y}|^3}\leq\frac{c}{|y|^4|x_0-\bar{y}|^4}.$$

Moreover, $|\mathbf{u}(y)|\leq[\mathbf{u}]_\alpha(1-|y|)^\alpha\leq[\mathbf{u}]_\alpha|x_0-\bar{y}|^\alpha$, since for any $x_0\in\Omega$ it holds that $1-|y|\leq|x_0-\bar{y}|$. With that in mind we derive

$$J_3^1\leq c|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha|x_1-x_2|\int_{\Omega\setminus B}\frac{1}{|y|^4|x_0-\bar{y}|^{4-\alpha}}.$$

The singularity in $y=0$ is in fact integrable. For any $\epsilon>0$ we have

$$\int_{\Omega\setminus B}\frac{1}{|y|^4|x_0-\bar{y}|^{4-\alpha}}\leq\int_{\Omega\setminus B\setminus B(0;\epsilon)}\frac{1}{|\epsilon|^4|x_0-y|^{4-\alpha}}$$
$$+\int_{\Omega\cap B(0;\epsilon)\setminus B}\frac{1}{|y|^\alpha|1-\epsilon|^{4-\alpha}}\leq c\epsilon^{-4}|x_1-x_2|^{-1+\alpha}+\frac{\epsilon^{3-\alpha}}{(1-\epsilon)^{4-\alpha}}. \tag{73}$$

Choose $\epsilon=1/4$. Combining last two estimate we have

$$J_3^1\leq c|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha|x_1-x_2|^\alpha. \tag{74}$$

By using (71) we deduce:

$$J_3^2\leq c|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha|x_1-x_2|\int_B\frac{1}{|y|^4|x_0-\bar{y}|^{4-\alpha}}. \tag{75}$$

For any $\epsilon > 0$ we have

$$\int_B \frac{1}{|y|^4|x_0-\bar{y}|^{4-\alpha}} \le \int_{B\setminus B(0;\epsilon)} \frac{1}{|\epsilon|^4|x_1-x_2||x_0-y|^{3-\alpha}}$$
$$+ \int_{B(0;\epsilon)\cap B} \frac{1}{|y|^\alpha|1-\epsilon|^{4-\alpha}} \le c\epsilon^{-4}|x_1-x_2|^{-1+\alpha} + \frac{\epsilon^{3-\alpha}}{(1-\epsilon)^{4-\alpha}}. \quad (76)$$

Choose $\epsilon = 1/4$. Combining last two estimate we have:

$$J_3^2 \le c|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha|x_1-x_2|^\alpha.$$

Consider now the case when $B \cap \partial\Omega \ne \emptyset$.

$$J_3(x_1) - J_3(x_2) = \int_{\Omega\cap B} \frac{k_{i,j}^2(x_1,y)}{|y|^3|x_1-\bar{y}|^3}\rho(y)u_i(y)u_j(y)$$
$$- \int_{\Omega\cap B} \frac{k_{i,j}^2(x_2,y)}{|y|^3|x_2-\bar{y}|^3}\rho(y)u_i(y)u_j(y)$$
$$+ \int_{\Omega\setminus B} \left\{ \frac{k_{i,j}^2(x_1,y)}{|y|^3|x_1-\bar{y}|^3} - \frac{k_{i,j}^2(x_2,y)}{|y|^3|x_2-\bar{y}|^3} \right\} \rho(y)u_i(y)u_j(y)$$
$$\triangleq J_{3,1} + J_{3,2} + J_{3,3}. \quad (77)$$

There is $x_0 \in \partial\Omega \cap B$ such that $|y - x_0| < 2|x_1 - \bar{y}|$, $\forall y \in \Omega \cap B$. And so,

$$J_{3,1} = \int_{\Omega\cap B} \frac{k_{i,j}^2(x_1,y)}{|y|^3|x_1-\bar{y}|^3}\rho(y)u_i(y)(u_j(y)-u_j(x_0))$$
$$\le c|\rho|_\infty|\mathbf{u}|_\infty < \mathbf{u} >_\alpha \int_{\Omega\cap B} |x_0-y|^{-3+\alpha}$$
$$\le c|\rho|_\infty|\mathbf{u}|_\infty[\mathbf{u}]_\alpha|x_1-x_2|^\alpha. \quad (78)$$

Term $J_{3,2}$ is estimated similarly. Term $J_{3,3}$ is estimated in the exactly the same way as J_3^1 since for all $y \in \Omega \setminus B$ inequality (71) holds.

To prove (69) we first write

$$J_1(x) = \int_\Omega \partial_{y_k}G_{1,2}(x,y)\rho(y)u_k(y) + \int_\Omega \partial_{y_k}G_{2,2}(x,y)\rho(y)u_k(y) \triangleq J_{1,1}+J_{1,2}.$$

Then, for $x_1\, x_2 \in \Omega$ we set $x_0 = (x_1 + x_2)/2$, and $B = B(x_0;, 2|x_1 -$

$x_2|$), we have

$$J_{1,1}(x_1) - J_{1,1}(x_2) = \int_{\Omega\cap B} \partial_{y_k} G_{1,2}(x_1, y)\rho(y)u_k(y)$$
$$+ \int_{\Omega\cap B} \partial_{y_k} G_{1,2}(x_2, y)\rho(y)u_k(y)$$
$$+ \int_{\Omega\setminus B} \{\partial_{y_k} G_{1,2}(x_1, y) - \partial_{y_k} G_{1,2}(x_2, y)\}\, \rho(y)u_k(y). \quad (79)$$

Notice, that uniformly in x, y,

$$|\nabla_y G_{1,1}| \le \frac{c}{|x-y|^2},\ |\nabla_x \nabla_y G_{1,1}| \le \frac{c}{|x-y|^3}.$$

Then, by the way of simple Hölder inequalities, it follows that $[J_{1,1}]_\gamma \le c|\rho\mathbf{u}|_p \le c|\rho|_\infty |\mathbf{u}|_p$, $p > 3$, $\gamma = 1 - \frac{1}{p}$.

□

Consider the following term from (30).

$$\int_\Omega \nabla_x G(x,y) \cdot \mathbf{F}_1(y) = -\frac{d}{dt}\int_\Omega \nabla_x G(x,y)\cdot \rho(y)\mathbf{u}(y)$$
$$+ u_j(x)\partial_{x_j} \int_\Omega \partial_{x_i} G^1(x,y)\rho(y)u_i(y) - \int_\Omega \partial_{x_i} G^1(x,y)\partial_{y_j}\left[\rho(y)u_i(y)u_j(y)\right]$$
$$+ u_j(x)\partial_{x_j} \int_\Omega \partial_{x_i} G^2(x,y)\rho(y)u_i(y) - \int_\Omega \partial_{x_i} G^2(x,y)\partial_{y_j}\left[\rho(y)u_i(y)u_j(y)\right]$$
$$= -\frac{d}{dt}\int_\Omega \nabla_x G\cdot\rho(y)\mathbf{u}(y) + \int_\Omega \partial_{y_j}\partial_{x_i} G^1(x,y)\left[\rho(y)u_i(y)(u_j(y) - u_j(x))\right]$$
$$+ u_j(x)\int_\Omega \partial_{x_j}\partial_{x_i} G^2(x,y)\rho(y)u_i(y) + \int_\Omega \partial_{y_j}\partial_{x_i} G^2(x,y)\rho(y)u_i(y)u_j(y)$$
$$= \frac{d}{dt}K_1(x) + K_2(x) + K_3(x) + K_4(x), \quad (80)$$

where we used notation (21). By repeating the arguments of the proof of the previous lemma we establish the following lemma.

Lemma 8. *There are $c_0 = c_0(\alpha)$, $c = c(\alpha, \beta)$ $c_1 = c_1(\gamma)$ such that*

$$|K_i|_\infty \le c_0|\rho|_\infty |\mathbf{u}|_\infty < \mathbf{u} >_\alpha, \quad (81)$$

$$[K_i]_\beta \le c_1|\rho|_\infty |\mathbf{u}|_\infty [\mathbf{u}]_\alpha, \quad \beta \in]0, \alpha[,\ i = 2, 3, 4. \quad (82)$$

$$|K_1|_\infty + [K_1]_\gamma \le c_1|\rho|_\infty |\mathbf{u}|_p, \quad \gamma = 1 - 3p^{-1},\ p > 3. \quad (83)$$

We also need following lemmata.

Lemma 9. *Let*

$$M(x) = \int_{\partial\Omega} \partial_{x_i}\partial_{n_y} G(x,\cdot)g(\cdot)\left\{u_i(\cdot) - u_i(x)\right\}\, dS, \quad x \in \Omega,$$

where $g \in C^{\omega}(\partial\Omega)$, $\omega > 1/2$. *Then*

$$|M|_\infty \le c(\omega)[\mathbf{u}]_{\frac{1}{2}}[g]_\omega, \quad [M]_{\frac{\omega-1}{2}} \le c(\omega)[\mathbf{u}]_{\frac{1}{2}}[g]_\omega.$$

Let

$$M_k(x) = \int_{\partial\Omega} \partial_{x_i}\left[\partial_{n_y} G(x,\cdot)(\cdot - x)\right] g(\cdot)\{u_i(\cdot) - u_i(x)\}\, dS, \quad x \in \Omega,\ k = 1..3.$$

Then,

$$|\nabla M_k|_\infty \le c(\omega)[\mathbf{u}]_{\frac{1}{2}}[g]_\omega, \quad x_1, x_2 \in \Omega.$$

Proof. We can write

$$M(x) = \int_{\partial\Omega} \partial_{x_i}\partial_{n_y} G(x,\cdot)\left\{g(\cdot) - g(x/|x|)\right\}\left\{u_i(\cdot) - u_i(x)\right\}\, dS,$$

for $x \neq 0$, since for any $x \in \Omega$, $\int_{\partial\Omega} \partial_{n_y} G(x,\cdot)\, dS = 1$. But $2|y - x/|x|| \le |y-x|$ for any $y \in \partial\Omega$, $x \in \Omega$. Thus, $|\left\{g(y) - g(x/|x|)\right\}\left\{u_i(y) - u_i(x)\right\}| \le c[\mathbf{u}]_{\frac{1}{2}}[g]_\omega |y-x|^{\frac{1+\omega}{2}}$. Also, $|\partial_{x_i}\partial_{n_y} G(x,y)| \le c|y-x|^{-3}$. So, the integrant in the definition of M has a singularity of type $|y-x|^\beta$, $\beta = -3 + \frac{1+\omega}{2} > -2$ a thus is integrable. The first assertion of lemma is proven. We omit the prove of the rest of the lemma as obvious, in the light of the above consideration. □

Lemma 10. *Let* $g(x) \in C^{\alpha}(\partial\Omega)$ *and*

$$M_1(x) = \int_{\partial\Omega} \partial_{n_y} G(x,y)g(y)\, dS, \quad x \in \Omega.$$

Then,

$$|M_1|_\infty \le |g|_{\infty,\partial\Omega}, \quad [M_1]_\alpha \le c(\alpha)[g]_{\alpha,\partial\Omega}, \quad 0 < \alpha < 1. \tag{84}$$

Let

$$M_{2,k}(x) = \int_{\partial\Omega} \partial_{n_y} G(x,y)(y_k - x_k)g(y)\, dS, \quad x \in \Omega,\ k = 1..3.$$

Then, $\forall\, \varepsilon > 0$ *there is* $c_\varepsilon = c_\varepsilon(\alpha)$ *such that*

$$|\nabla M_{2,k}|_\infty \le \varepsilon[g]_{\alpha,\partial\Omega} + c_\varepsilon |g|_{\infty,\partial\Omega}, \quad \alpha \in]0,1[. \tag{85}$$

Lemma 11. *Let* $L_i(x) = \int_\Omega \partial_{y_i} G(x,\cdot)(P(\cdot) - \overline{P})$, $i = 1..3$. *Then,* $\forall\, \varepsilon > 0$ *there is* $c_\varepsilon = c_\varepsilon(\alpha)$ *such that*

$$|L_i(x_1) - L_i(x_2)|_\infty \leq \varepsilon \langle P\rangle_{\alpha,\partial\Omega}|x_1 - x_2| + c_\varepsilon|P - \overline{P}|_\infty|x_1 - x_2|, \quad (86)$$

$\forall\, x_1 \in \partial\Omega$, $x_2 \in \Omega = B(0;1)$.

Proof. The proof is an easy adaptation of the lemma 8.1 in [7]. □

2.5 Energy estimates

Multiplying equations (2) by $\mathbf{u}$, using (1) we obtain:

$$\frac{d}{dt}\int \rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2/2 + \int (\lambda + 2\mu)|\mathrm{div}\,\mathbf{u}(t,\cdot)|^2 + \mu|\,\mathrm{curl}\,\mathbf{u}(t,\cdot)|^2$$
$$- \int (P(t,\cdot) - \overline{P})\mathrm{div}\,\mathbf{u} \leq 0. \quad (87)$$

With notation $\Psi(\rho) = \rho\int_{\bar\rho}^{\rho} s^{-2}(P(\rho) - \overline{P}) \geq 0$, and $E(t) = \int \rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2 + 2\Psi(\rho(t,\cdot))$ we obtain that

$$E(t) \leq E(0), \quad t > 0.$$

Let us consider equations (2), divide them by ρ and take operators div and curl of the result. We get:

$$\frac{d}{dt}\mathrm{div}\,\mathbf{u} + \mathrm{div}\,((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathrm{div}\,\mathbf{u}$$
$$- \mathrm{div}\,\left[\rho^{-1}(\lambda + 2\mu)\nabla\mathrm{div}\,\mathbf{u} - \rho^{-1}\mu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}\right] = 0, \quad (88)$$

$$\frac{d}{dt}\,\mathrm{curl}\,\mathbf{u} + \mathrm{curl}\,((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)\,\mathrm{curl}\,\mathbf{u}$$
$$- \mathrm{curl}\,\left[\rho^{-1}(\lambda + 2\mu)\nabla\mathrm{div}\,\mathbf{u} - \rho^{-1}\mu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}\right] = 0. \quad (89)$$

Using the relation $F = (\lambda + 2\mu)\mathrm{div}\,\mathbf{u} - (P - \overline{P})$, we multiply the first equation by F, second (dot product) by $\mu\,\mathrm{curl}\,\mathbf{u}$, add them and integrate over Ω After carrying out the integration by parts on the principal part we obtain:

$$\frac{d}{dt}\frac{1}{2}\int \frac{F^2}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}|^2 + \int \rho^{-1}|\mathbf{F}_1|^2$$
$$= \frac{1}{2}\int (\frac{|F|^2}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}|^2)\operatorname{div}\mathbf{u}$$
$$+ \Big\{\int (\operatorname{div}((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)F)\operatorname{div}\mathbf{u}$$
$$+ \int (\operatorname{curl}((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)\operatorname{curl}\mathbf{u})\cdot\mu\operatorname{curl}\mathbf{u}\Big\}$$
$$+ \int \frac{A\rho\operatorname{div}\mathbf{u}F}{\lambda+2\mu} \triangleq J_1 + J_2 + J_3, \quad (90)$$

where $\mathbf{F}_1 = (\lambda+2\mu)\nabla\operatorname{div}\mathbf{u} + \mu\operatorname{curl}\operatorname{curl}\mathbf{u})$. Both terms,

$$|J_1|(|J_2|) \lesssim (\lambda+2\mu)|\nabla\mathbf{u}|_3^3 + \frac{|P-\overline{P}|_3^3}{(\lambda+2\mu)^3} \quad (91)$$

Note, that $\lambda + 2\mu > \frac{4\mu}{3}$. On the other hand,

$$J_3 \le \frac{A\hat{\rho}}{\lambda+2\mu}\int (\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2 - \frac{A}{\lambda+2\mu}\int \rho(P-\overline{P})\operatorname{div}\mathbf{u}. \quad (92)$$

Let us estimate $|\nabla\mathbf{u}|_3^3$. By lemma 1 we have $|\nabla\mathbf{u}|_3^3 \lesssim |\nabla\mathbf{u}|_2^{\frac{3}{2}}|\nabla\mathbf{u}|_6^{\frac{3}{2}}$ and thus by lemma 5 we get (for any $\epsilon > 0$):

$$|\nabla\mathbf{u}|_3^3 \lesssim \epsilon|\mathbf{F}_1|_2^2 + \epsilon\langle P\rangle_{\alpha,\partial\Omega}^2 + c_\epsilon|\nabla\mathbf{u}|_2^6. \quad (93)$$

A simple energy estimate of the Lamé equations (19) and (20) leads to the following estimate.

$$\int (\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2 + \mu|\operatorname{curl}\mathbf{u}|^2 \le c|\mathbf{F}_1|_2^2 + c|P-\overline{P}|_2^2,$$

where $c = c(\lambda,\mu)$. Also,

$$\int \frac{F^2}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}|^2 \le c\int (\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2 + \mu|\operatorname{curl}\mathbf{u}|^2 + c|P-\overline{P}|_2^2.$$

We use the last two inequalities and also estimates (91), (92), (93) in (90) and add the result to inequality (87) multiplied by $\frac{2A\hat{\rho}}{\lambda+2\mu}$ to get:

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int \rho|\mathbf{u}|^2 + \frac{F^2}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}|^2\right\} \\
&\qquad + \left\{\int \rho|\mathbf{u}|^2 + \frac{F^2}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}|^2\right\} + |\mathbf{F}_1|_2^2 + |\nabla\mathbf{u}|_2^2 \\
&\quad - c^0\int(2\hat{\rho}-\rho)(P-\overline{P})\operatorname{div}\mathbf{u} \le c_\epsilon|\nabla\mathbf{u}|_2^6 + c_\epsilon|P-\overline{P}|_2^2 + \epsilon\langle P\rangle_{\alpha,\partial\Omega}^2,
\end{aligned}
$$

with $c^0 = c(\lambda,\mu,A,\bar{\rho})$. Note, that we used the fact that $|\rho\mathbf{u}|_2 \le c|\nabla\mathbf{u}|_2$. Now, if we set $\Phi(\rho) = c^0\rho\int_{\bar{\rho}}^{\rho} s^{-2}(2\hat{\rho}-s)(P(s)-\overline{P})\,ds$. Then, $-c^0\int(2\hat{\rho}-\rho)(P-\overline{P})\operatorname{div}\mathbf{u} = \frac{d}{dt}\int\Phi(\rho)$. Under hypothesis H_1, $\Phi \sim (\rho-\bar{\rho})^2$ which leads to the inequality $\int\Phi(\rho) \le c|P-\overline{P}|_2^2$ and we obtain:

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int \rho|\mathbf{u}|^2 + \frac{F^2}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}|^2 + \Phi(\rho)\right\} \\
&\quad + \left\{\int \rho|\mathbf{u}|^2 + \frac{F^2}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}|^2 + \Phi(\rho)\right\} + |\mathbf{F}_1|_2^2 + |\nabla\mathbf{u}|_2^2 \\
&\qquad\qquad \le c_\epsilon|\nabla\mathbf{u}|_2^6 + c_\epsilon|P-\overline{P}|_2^2 + \epsilon\langle P\rangle_{\alpha,\partial\Omega}^2. \quad (94)
\end{aligned}
$$

Let us make the following assumption.

Hypothesis H_2 :
$\sup_{t\in]0,\infty[}|\nabla\mathbf{u}(t,\cdot)|_2^4 \le (2c_\epsilon)^{-1}$.

Let

$$
Y(t) = \int \rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2 + \frac{F^2(t,\cdot)}{\lambda+2\mu} + \mu|\operatorname{curl}\mathbf{u}(t,\cdot)|^2 + \Phi(\rho(t,\cdot)).
$$

Then, it follows from (87) and (94) that

$$
\begin{aligned}
Y(t) \le Y(0)e^{-mt} + c_\epsilon\int_0^t e^{m(\tau-t)}|P(\tau,\cdot)-\overline{P}|_2^2 \\
+ \epsilon\int_0^t e^{m(\tau-t)}\langle P\rangle_{\alpha,\partial\Omega}^2(\tau), \quad (95)
\end{aligned}
$$

for $m=\frac{1}{2}$ and for any $\bar{m}>0$:

$$\int_0^t e^{-\bar{m}(t-\tau)}(|\mathbf{F}_1(\tau,\cdot)|_2^2+|\nabla\mathbf{u}(\tau,\cdot)|_2^2)\lesssim Y(0)$$
$$+c_\epsilon\int_0^t e^{-\min\{m,\bar{m}\}(t-\tau)}|P(\tau,\cdot)-\overline{P}|_2^2+\epsilon\int_0^t e^{-\min\{m,\bar{m}\}(t-\tau)}\langle P\rangle_{\alpha,\partial\Omega}^2(\tau) \tag{96}$$

2.6 Uniform estimates on density

Let us fix $x_1\in\partial\Omega$, $x_2\in\Omega$ and consider two flow trajectories; $X(t,\tau;x_1)$, $X(t,\tau;x_2)$ for $\tau\in[0,t]$. Let $\rho^i(\tau)$, $i=1,2$, be the restriction of $\rho(t,x)$ to the trajectory that passes through x_i at $\tau=t$, and $\Delta\rho=\rho^1-\rho^2$. We will use superscript $i=1,2$ and Δ to denote the corresponding quantities for other functions as well. Finally, let $\tilde{\rho}\in[\rho^1,\rho^2]$, be such that $\rho^1-\rho^2=\tilde{\rho}(\log\rho^1-\log\rho^2)$. Consider equation (1). It can be written in the following form, see (43).

$$(\lambda+2\mu)\frac{d}{dt}((\tilde{\rho})^{-1}\Delta\rho)+A\Delta\rho=-\frac{d}{dt}\Delta h-\Delta l. \tag{97}$$

Let $\omega(t)=(\lambda+2\mu)^{-1}\int_0^t\tilde{\rho}\geq\check{\rho}t$. Then,

$$\begin{aligned}(\lambda+2\mu)\tilde{\rho}^{-1}(t)\Delta\rho(t)=(\lambda+2\mu)&e^{-A\omega(t)}\Delta\log\rho_0\\ &-\Delta h(t)+e^{-A\omega(t)}\Delta h(0)\\ +A(\lambda+2\mu)^{-1}&\int_0^t\Delta h(\cdot)\tilde{\rho}(\cdot)e^{-A(\omega(t)-\omega(\tau))}\\ &-\int_0^t\Delta l(\cdot)e^{-A(\omega(t)-\omega(\tau))}.\end{aligned} \tag{98}$$

Let $\alpha\in]0,\frac{1}{2}[$. Assume

Hypothesis H_3:
$\alpha cX(t)\leq\frac{\check{\rho}}{2(\lambda+2\mu)},\quad t>0,$
$A_0^\alpha\leq\frac{3}{2},$

where c, A_0 and X as in (54) which leads to the inequality:

$$d(\tau)^\alpha\leq\frac{3}{2}d(0)^\alpha e^{\frac{\check{\rho}}{2(\lambda+2\mu)}\tau},\quad\tau\in[0,t]. \tag{99}$$

Note, that by (44), Poincaré-Sobolev inequality and Hypothesis H_2 we have for $t > 0$ that

$$[h(t,\cdot)]_\alpha \lesssim |\rho(t,\cdot)\mathbf{u}(t,\cdot)|_6 \\ \lesssim |\sqrt{\rho(t,\cdot)}\mathbf{u}(t,\cdot)|_2^{\frac{3}{2}} |\nabla \mathbf{u}(t,\cdot)|_2^{\frac{3}{2}} \leq c_\epsilon E(0). \quad (100)$$

We divide the equation (98) by $d(0)^\alpha$ and use the estimate for $d(\cdot)$ given by (99) and (100) to derive the next inequality.

$$(\lambda + 2\mu)\hat{\rho}^{-1}\langle \rho(t,\cdot)\rangle_{\alpha,\partial\Omega} \leq c(\lambda + 2\mu)\langle \rho_0 \rangle_{\alpha,\partial\Omega} \\ + c_\epsilon E(0) + \frac{3}{2}\int_0^t [l(\tau,\cdot)]_\alpha e^{-\frac{A\check{\rho}}{2(\lambda+2\mu)}(t-\tau)}. \quad (101)$$

Now, we use (45) and (96) with $\bar{m} = A\check{\rho}(2(\lambda + 2\mu))^{-1}$ to get the inequality

$$\sup_{\tau\in]0,t[} \hat{\rho}^{-1}\langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega} \leq c\langle \rho_0\rangle_{\alpha,\partial\Omega} \\ + c_e(E(0) + Y(0)) + c_e \sup_{\tau\in]0,t[} |\rho(\tau,\cdot) - \bar{\rho}| \sup_{\tau\in]0,t[} \langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega} \\ + \epsilon \sup_{\tau\in]0,t[} \langle P(\tau,\cdot)\rangle^2_{\alpha,\partial\Omega} + \frac{3c_1(\alpha)\alpha_4}{\check{\rho}} \sup_{\tau\in]0,t[} \langle P(\tau,\cdot)\rangle_{\alpha,\partial\Omega}. \quad (102)$$

It follows that if

$$\alpha_4 = \frac{2\mu}{5\lambda + 7\mu} < \frac{\check{\rho}}{3c_1(\alpha)\hat{\rho}},$$

$|\rho - \bar{\rho}|_\infty$ and ϵ are sufficiently small (depending on λ, μ, α), then

$$\sup_{\tau\in]0,t[} \langle \rho(\tau,\cdot)\rangle_{\alpha,\partial\Omega} \leq c\langle \rho_0\rangle_{\alpha,\partial\Omega} + c\,(E(0) + Y(0))\,. \quad (103)$$

In a completely analogous, but easier way we obtain an estimate on $\operatorname{osc}\rho$.

$$\sup_{\tau\in]0,t[} |\rho(\tau,\cdot) - \bar{\rho}|_\infty \leq c|\rho_0 - \bar{\rho}|_\infty + c_\delta\,(E(0) + Y(0)) \\ + \delta \sup_{\tau\in]0,t[} \langle \rho(\tau,\cdot)\rangle_{\alpha,\partial\Omega},\ \delta > 0. \quad (104)$$

It follows from the last two inequalities that $\sup_{\tau\in]0,t[} |\rho(\tau,\cdot) - \bar{\rho}|_\infty$ as well as $\sup_{\tau\in]0,t[}\langle \rho(\tau,\cdot)\rangle_{\alpha,\partial\Omega}$ can bounded in terms of initial data only and independently of t. Thus, initial data can be restricted in such a way to meet all Hypotheses H_1 - H_3.

Finally, we list all the information that we obtained by the way of *a priori* estimates. The following norms and seminorms are bounded in terms of initial data measured by $\langle \rho_0 \rangle_{\alpha,\partial\Omega} + |\nabla \mathbf{u}_0|_2$ and $T > 0$.

$$\sup_t \{ |\rho(t,\cdot)|_\infty, \langle \rho(t,\cdot) \rangle_{\alpha,\partial\Omega}, |\nabla \mathbf{u}(t,\cdot)|_2 \},$$

$$|\mathbf{u}_t|_{L^2((0,T):W^{-1,2}(\Omega))}, \int_0^T [\mathbf{u}(t,\cdot)]_{\frac{1}{2}}, \quad T > 0. \tag{105}$$

3 Proof of the existence

Consider now the sequence of initial data of the problem

$$\rho_0^n, \mathbf{u}_0^n, \in C^\infty(\bar{\Omega}) \times C_0^\infty(\bar{\Omega})^3,$$

which approximates the given initial data in the space

$$L^4 \times \left(W_0^{1,2} \right)^3.$$

Moreover, we require that

$$\hat{\rho} > \rho_0^n > \check{\rho} > 0,$$

and the assumptions on initial data under which (103) was obtained is satisfied. Such a sequence, clearly exists. For ρ_0 we take $\rho_0^n(x) = \rho_0(x) * \omega_{n^{-1}}(x)$, where ω_ϵ is the standard mollifier. As for the velocity it holds that

$$W_0^{1,2}(\Omega) = \text{ closure of } C_0^\infty(\Omega) \text{ in } W^{1,2} \text{ norm.}$$

Accordingly, let $\rho^n, \mathbf{u}^n$ be the sequence of smooth solutions of the problem with $\rho_0^n, \mathbf{u}_0^n$ as the data. The solutions exist locally in time by the results in [12] and can be extended globally, since the estimates (105) hold, independently of the time interval of existence of such solutions. Thus, there is a subsequence, still labelled by n, and ρ, $\mathbf{u} \in L^\infty(\mathbb{R}_+ \times \Omega) \times L^2\left(\mathbb{R}_+; W_0^{1,2}(\Omega)^3\right)$, such that

$$\rho^n \to \rho, \quad * - weakly\, L^\infty(\mathbb{R}_+ \times \Omega), \tag{106}$$

$$\mathbf{u}^n \to \mathbf{u}, \quad weakly\, L^2\left(\mathbb{R}_+; W_0^{1,2}(\Omega)^3\right), \tag{107}$$

and estimates (105) hold. In fact, the subsequence can be chosen in such a way that

$$\mathbf{u}^n \to \mathbf{u}, \quad L^2\left((0,T); L^2(\Omega)^3\right), \, T > 0.$$

Indeed, this is the consequence of estimates (105), see [13](Theorem 2.1, Chap.3). Using lemma 1 and (105) we deduce that

$$\mathbf{u}^n \to \mathbf{u}, \quad L^8\left((0,T); L^4(\Omega)^3\right), \, \forall T > 0. \tag{108}$$

Moreover, it follows from the result of DiPerna - P.L.Lions, see [5](Theorem 2.4) and estimates (105) that

$$\rho \in C\left([0,T); L^p(\Omega)\right), \, 1 < p < \infty, \, T.$$

Also, (105) imply that $\mathbf{u} \in C\left([0,T); L^2(\Omega)\right)^3$. We take the limit in the weak formulation of the problem, satisfied by ρ^n, $\mathbf{u}^n$. Using (106)-(108) is clearly enough to pass to the limit in the equations (9) and (10).

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