Practice Problems Math 235 Fall 2011 for Final Exam

- Please use correct notation when writing matrices and vectors.
- Explain how you arrived at your answers, please, and show your algebraic calculations. Use the back of the preceding page if necessary.
- Please justify your statements. Unsubstantiated answers receive no credit.

1. False or True. (Please justify any "False" answer with a counter-example and any "True" answer with logical reasoning.)

   1a: Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and assume that $\ker(L) = \{0\}$. Then for each $y \in \mathrm{im}(L)$ there is a unique solution to the equation $L(x) = y$.

   1b: The rank of a linear map $\mathbb{R}^{2011} \to \mathbb{R}^{2010}$ is at most 2010.

   1c: There is a linear map from $\mathbb{R}^{2011} \to \mathbb{R}^{2010}$ whose kernel is $\{0\}$.

   1d: Any set of 2010 vectors in $\mathbb{R}^{2011}$ must be linearly independent.

   1e: Reflection about a line in the plane is a linear map represented by a matrix of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where a, b are real numbers satisfying $a^2 + b^2 = 1$.

   1f: Suppose V is the vector space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. The map $F : V \to V : f(x) \mapsto f''(x) - 3f'(x) + 6f(x) - 9$ is linear.

   1g: Every real 3×3 matrix has a real eigenvalue and real eigenvector.

   1h: The matrices
   $$\begin{pmatrix} -4 & 2 \\ -6 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} -7/2 & 3/2 \\ -9/2 & 5/2 \end{pmatrix}$$
   are similar.

   1i: The matrices
   $$\begin{pmatrix} -5/4 & -3/4 \\ -9/4 & 1/4 \end{pmatrix} \text{ and } \begin{pmatrix} -7/2 & 3/2 \\ -9/2 & 5/2 \end{pmatrix}$$
   are similar.

2. Let T be the linear transformation from P_2 to P_2 given by

$$f(x) \mapsto f'' - 2f.$$

Find the matrix of T with respect to the basis $\{1, x-1, (x-1)^2\}$.

3. Let P_2 be the vector space of quadratic polynomials with standard basis $S = \{1, t, t^2\}$, and let $T : P_2 \to P_2 : p(t) \mapsto p'(t) + p(t)$.

   (a) Verify that T is a linear map.

   (b) Compute the matrix $[T]_{SS}$ for T with respect to the basis S.

   (c) Is T an isomorphism? (Why or why not?)

   (d) Find all polynomials $p(t)$ such that $T(p(t)) = 1 + t + t^2$.

4. Let $a = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and consider the map from $\mathbb{R}^3$ to itself defined by cross product: $v \mapsto v \times a$.

   (a) Show that this map is linear.

   (b) Compute the matrix for this map with respect to the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

   (c) Find a basis for its image and kernel.

5. Let $v = (1, -1, 1)$.

   (a) Write the 3×3 matrix of orthogonal projection onto $\mathrm{span}(v)$ with respect to the standard basis E of $\mathbb{R}^3$.

   (b) Write the 3×3 matrix of reflection in the mirror plane orthogonal to v in this basis E.

6. Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis, and let $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$ be another basis of $\mathbb{R}^2$.

   (a) Suppose $v \in \mathbb{R}^2$ has coordinates $[v]_S = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ with respect to the standard basis S. What are its coordinates $[v]_B$ with respect to B?

   (b) If a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix $[L]_{SS} = \begin{pmatrix} 9 & -8 \\ 10 & -9 \end{pmatrix}$ in the standard basis S, what is its matrix $[L]_{BB}$ in the basis B?

7. Compute the determinant of the matrix $B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 5 & 6 & 0 \\ 0 & 7 & 8 & 0 \\ 3 & 0 & 0 & 4 \end{pmatrix}$.

   Find $\det(B^{235})$. If B is invertible, find $\det(B^{-1})$.

8. Let $C = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

   (a) Compute the characteristic polynomial of C.

   (b) Find the eigenvalues of C and an eigenvector for each eigenvalue.

   (c) Is C diagonalizable, that is, can we find an invertible matrix E so that $D = E^{-1}CE$ is diagonal? If so, find such matrices E and D; if not, explain why not.

9. Let $C = \begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix}$.

   (a) Compute the characteristic polynomial of C.

   (b) Find the eigenvalues of C and an eigenvector for each eigenvalue.

   (c) Is C diagonalizable, that is, can we find an invertible matrix E so that $D = E^{-1}CE$ is diagonal? If so, find such matrices E and D; if not, explain why not.

10. Let
$$A = \begin{pmatrix} 1/3 & 10/3 \\ -4/3 & 5/3 \end{pmatrix}.$$

   (a) Compute the characteristic poynomial of A.

   (b) Find the eigenvalues and eigenvectors of A.

   (c) The eigenvalues are not real. Find a basis B of $\mathbb{R}^2$ so that the matrix of A with respect to the basis B is of the form
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

11. Let
$$Z = \begin{pmatrix} -8 & 15 \\ -6 & 10 \end{pmatrix}.$$

   (a) Find the eigenvalues for Z and for each eigenvalue find an eigenvector.

   (b) There is a matrix S so that matrix $S^{-1}ZS$ is a matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$. What are a, b? What is S?

12. Let v be an eigenvector of an $n \times n$ matrix B with eigenvalue 3. Explain why v is an eigenvector of $C = B^2 - 2B - I_n$. What is its eigenvalue of v as an eigenvector of C.

13. The matrix

$$A = \begin{pmatrix} -5/2 & 3/2 \\ -9 & 5 \end{pmatrix}$$

has eigenvectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(a) What are the eigenvalues for these eigenvectors?

(b) Find a closed form for A^n. This should be the product of three explicitly given matrices. You need not compute the inverse of a matrix or multiply out the three matrices.

14. Consider the complex number $z = .6 - .7i$. Represent the powers $z^2, z^3, \cdots, z^k, \cdots$ in the complex plane and explain their long-term behavior as k gets larger and larger.