Math 235 Suggestions for Final

1. For the matrice below find the characteristic equation, the eigenvalue(s), and for each eigenvalue find an eigenvector. Is the matrix are diagonalizable over the real numbers?

   (a) $A = \begin{pmatrix} 9/5 & -2/5 \\ -2/5 & 6/5 \end{pmatrix}$

   Answer: The characteristic equation is $\lambda^2 - 3\lambda + 2$. Eigenvalue 1 and eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and Eigenvalue 2 and eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ Yes, this is diagonalizable since the sum of the dimensions of the eigen spaces is 2 and this is a 2×2 matrix.

2. Let
$$A = \begin{pmatrix} 5/3 & 1/3 \\ 2/3 & 4/3 \end{pmatrix}.$$
The eigenvectors of A are
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
and the eigenvalues are $2, 1$. Find a matrix B so that
$$BAB^{-1}$$
is diagonal.

3. Let
$$M = \begin{pmatrix} 5/2 & -1/4 \\ -1 & 5/2 \end{pmatrix}.$$
Write M^{2013} as the product of three matrices. It is not necessary to multiply out this product. You may have an entry of the form a^{2013} for some number a.

   Answer: The eigenvalues of M are $2, 3$ and the eigenvectors are
$$u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$
Let $B = \{u, v\}$ be a basis of $\mathbb{R}^2$ and let A be the matrix that converts B-coordinates to standard coordinates, so
$$A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$
So
$$A^{-1} = (-1/4) \begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix}$$

is the inverse of A. Then

$$M = A \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} A^{-1}.$$

Hence

$$M^{2013} = A \begin{pmatrix} 2^{2013} & 0 \\ 0 & 3^{2013} \end{pmatrix} A^{-1}$$

4. Define the following terms:
   (a) Subspace
   (b) Linear map
   (c) Kernel
   (d) Image
   (e) Rank
   (f) Dimension
   (g) Span

5. Prove the following statements:
   (a) The kernel of a linear map is a subspace.
   (b) The image of a linear map is a subspace.
   (c) Let $F : V \to W$ be a linear map of vector spaces and let $a \in V$ be a solution to the equation $F(x) = b$. Show that every solution to the equation $F(x) = b$ is of the form $a + v$ where $v \in ker(F)$.

6. State the rank- nullity theorem and define each of the terms.

7. True or False. You must explain your reason for choosing true or false.
   (a) Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and assume that $\ker(L) = \{0\}$. Assume that $n < m$. Then for each $y \in \mathrm{im}(L)$ there is a unique solution to the equation $L(x) = y$.
   (b) Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and assume that $\ker(L) = \{0\}$. Assume that $n = m$. Then for each $y \in \mathrm{im}(L)$ there is a unique solution to the equation $L(x) = y$.
   (c) Any set of 2012 vectors in $\mathbb{R}^{2013}$ must be linearly independent.
   (d) Suppose V is the vector space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. The map $F : V \to V : f(x) \mapsto f''(x) - 3f'(x) + 6f(x)^2$ is linear.

8. Let U denote the set of upper triangular 2×2 matrices. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Consider the map

$$\mathbf{A} : U \to U, \quad X \mapsto AX.$$

a: Show this is linear.
b: Exhibit a basis of U.
c: Find the matrix of $\mathbf{A}$ with respect to this basis.

9. Define

$$D : P_2 \to P_2, \quad f \mapsto xf' - 3f.$$

Find the basis of D with respect to the basis $\{1, x, x^2\}$. Using the matrix find the kernel of D.

10. For each the following two matrices find a basis of each of the eigenspaces. For each matrix determine whether it is diagonalizable. Show work.

(a)

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

Answer: the eigenvalues are $-2, 1$. A basis for the eigenspace for -2 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. It is one dimensional.

A basis for the eigenvalue 1 is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The sum of the diemensions of the eigenspaces is 2. This is strictly less than 3 so the matrix is NOT diagonalizable.

(b)

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 2 & -3 \\ 0 & 0 & -1 \end{pmatrix}$$

Answer: The eigenvlaues are $-1, 2$. The eigenspace of -1 is two dimensional. There are many bases. Here is one: $\begin{pmatrix} 1, \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

The eigenspace for eigenvalue 2 is one dimensional. A basis is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Since the sum of the dimensions of the eigenspaces is 3 (this is a 3×3 matrix), this is diagonalizable.