# 235 Final exam review questions 

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(1) Let $A$ be an $n \times n$ matrix and $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}, T(\mathbf{x})=A \mathbf{x}$ the linear transformation with matrix $A$. What does it mean to say that a vector $\mathbf{v} \in \mathbb{R}^{n}$ is an eigenvector of $A$ (or $T$ ) with eigenvalue $\lambda$ ?
(2) Arguing geometrically, describe the eigenvalues and eigenvectors of the following linear transformations.
(a) $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ given by reflection in the line $y=2 x$.
(b) $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ given by orthogonal projection onto the line $y=3 x$.
(c) $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ the horizontal shear given by $T(\mathbf{x})=\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right) \mathbf{x}$.
(3) Let $A$ be an $n \times n$ matrix. Here is the strategy to find the eigenvalues and eigenvectors of $A$ :
(a) Solve the characteristic equation $\operatorname{det}(A-\lambda I)=0$ to find the eigenvalues.
(b) For each eigenvalue $\lambda$ solve the equation $(A-\lambda I) \mathbf{v}=\mathbf{0}$ to find the eigenvectors $\mathbf{v}$ with eigenvalue $\lambda$.
[Why does this work? The equation $(A-\lambda I) \mathbf{v}=\mathbf{0}$ is obtained from the equation $A \mathbf{v}=\lambda \mathbf{v}$ by rearranging the terms. This equation has a nonzero solution $\mathbf{v} \in \mathbb{R}^{n}$ exactly when $(A-\lambda I)$ is not invertible, equivalently $\operatorname{det}(A-\lambda I)=0$.]

The function $\operatorname{det}(A-\lambda I)$ is a polynomial of degree $n$ in the variable $\lambda$. In particular if $n=2$ and $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ then

$$
\begin{gathered}
\operatorname{det}(A-\lambda I)=\operatorname{det}\left(\begin{array}{cc}
a-\lambda & b \\
c & d-\lambda
\end{array}\right) \\
=(a-\lambda)(d-\lambda)-b c=\lambda^{2}-(a+d) \lambda+(a d-b c)
\end{gathered}
$$

and we can solve the characteristic equation using the quadratic formula. If $n=3$ we can determine the polynomial $\operatorname{det}(A-\lambda I)$ by computing the determinant using either Sarrus' rule or expansion along a row or column.
(4) For each of the following matrices, find all the eigenvalues and eigenvectors.
(a) $\left(\begin{array}{ll}2 & 1 \\ 0 & 3\end{array}\right)$
(b) $\left(\begin{array}{ll}2 & 1 \\ 2 & 3\end{array}\right)$
(c) $\left(\begin{array}{cc}1 & 1 \\ -1 & 3\end{array}\right)$
(d) $\left(\begin{array}{lll}1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1\end{array}\right)$
(e) $\left(\begin{array}{ccc}1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1\end{array}\right)$
(f) $\left(\begin{array}{lll}3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4\end{array}\right)$
(5) Let

$$
A=\left(\begin{array}{ccc}
0 & -1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{array}\right)
$$

The linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}, T(\mathbf{x})=A \mathbf{x}$ is given by rotation about a line $L$ through some angle $\theta$. Find the line $L$.
[Hint: A vector $\mathbf{v}$ in the direction of $L$ is an eigenvector of $A$ (why?). What is the corresponding eigenvalue?]
(6) Let $A$ be an $n \times n$ matrix. We say $A$ is diagonalizable if there is a basis $\mathcal{B}$ of $\mathbb{R}^{n}$ consisting of eigenvectors of $A$. In this case, let $\mathcal{B}=\left(\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}\right)$ be the basis of eigenvectors, with eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$. Then the $\mathcal{B}$-matrix of the transformation $T(\mathbf{x})=A \mathbf{x}$ is the diagonal matrix $D$ with diagonal entries the eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ (why?). Equivalently, writing $S$ for the matrix with columns the vectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$, we have

$$
A=S D S^{-1}
$$

We can determine whether $A$ is diagonalizable as follows: for each eigenvalue $\lambda$, find a basis of the eigenspace $E_{\lambda}=\operatorname{ker}(A-\lambda I)$ (the subspace of $\mathbb{R}^{n}$ consisting of all the eigenvectors with eigenvalue $\lambda$ together with the zero vector). Now combine the bases of all the eigenspaces. These vectors are linearly independent. If there are $n$ vectors, then they form a basis $\mathcal{B}$ of $\mathbb{R}^{n}$ and $A$ is diagonalizable, otherwise $A$ is not diagonalizable.
(7) For each of the matrices $A$ of Q 4 , determine whether $A$ is diagonalizable. If $A$ is diagonalizable identify a basis $\mathcal{B}$ of $\mathbb{R}^{n}$ consisting of eigenvectors of $A$ and write down the $\mathcal{B}$-matrix of the linear transformation $T(\mathbf{x})=A \mathbf{x}$.
(8) For which values of $a$ and $b$ is the matrix $A=\left(\begin{array}{ll}2 & a \\ 0 & b\end{array}\right)$ diagonalizable?
(9) If $A$ is diagonalizable we can compute an explicit formula for powers of $A$ as follows: Write $A=S D S^{-1}$ as above where $D$ is the diagonal matrix with diagonal entries the eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$. Then for any positive integer $k$ we have

$$
A^{k}=S D^{k} S^{-1}
$$

(why?) and $D^{k}$ is the diagonal matrix with diagonal entries $\lambda_{1}^{k}, \ldots, \lambda_{n}^{k}$.
(10) For the matrices $A$ of $\mathrm{Q} 4(\mathrm{a})$ and (b) compute a formula for $A^{k}$.
(11) Let $W \subset \mathbb{R}^{3}$ be the subspace with basis $\mathcal{B}=\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ where

$$
\mathbf{v}_{1}=\left(\begin{array}{c}
2 \\
-1 \\
2
\end{array}\right), \mathbf{v}_{2}=\left(\begin{array}{c}
1 \\
-1 \\
3
\end{array}\right)
$$

(a) Using the Gram-Schmidt process, find an orthonormal basis $\mathcal{C}=$ $\left(\mathbf{u}_{1}, \mathbf{u}_{2}\right)$ for $W$.
(b) Let $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ be the linear transformation given by orthogonal projection onto $W$. Write down a formula for $T(\mathbf{x})$ in terms of $\mathbf{u}_{1}$ and $\mathbf{u}_{2}$, and use it to compute $T\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$.
(12) Let $W \subset \mathbb{R}^{4}$ be the subspace with basis $\mathcal{B}=\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ where

$$
\mathbf{v}_{1}=\left(\begin{array}{l}
1 \\
1 \\
1 \\
1
\end{array}\right), \mathbf{v}_{2}=\left(\begin{array}{l}
3 \\
1 \\
3 \\
1
\end{array}\right)
$$

(a) Using the Gram-Schmidt process, find an orthonormal basis $\mathcal{C}=$ $\left(\mathbf{u}_{1}, \mathbf{u}_{2}\right)$ of $W$.
(b) Let $T: \mathbb{R}^{4} \rightarrow \mathbb{R}^{4}$ be orthogonal projection onto $W$. Compute $T\left(\begin{array}{l}3 \\ 5 \\ 1 \\ 3\end{array}\right)$.
(13) Let

$$
\mathbf{u}_{1}=\frac{1}{9}\left(\begin{array}{c}
4 \\
-1 \\
-8
\end{array}\right), \mathbf{u}_{2}=\frac{1}{9}\left(\begin{array}{c}
-7 \\
4 \\
-4
\end{array}\right), \mathbf{u}_{3}=\frac{1}{9}\left(\begin{array}{l}
4 \\
8 \\
1
\end{array}\right) .
$$

(a) Show that $\mathcal{B}=\left(\mathbf{u}_{1}, \mathbf{u}_{2}, \mathbf{u}_{3}\right)$ is an orthonormal basis of $\mathbb{R}^{3}$.
(b) Let $\mathbf{v}=\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)$. Using part (a) or otherwise, compute the $\mathcal{B}$-coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$.
(14) Find all solutions of the following system of linear equations. Write your answer as a linear combination of vectors in $\mathbb{R}^{5}$.

$$
\begin{array}{r}
x_{1}-x_{2}+x_{3}+x_{2}+2 x_{5}=1 \\
2 x_{1}-x_{2}+4 x_{3}+x_{4}+3 x_{5}=3 \\
-x_{1}+3 x_{2}+3 x_{3}+5 x_{4}-x_{5}=7
\end{array}
$$

(15) Let $V$ be a linear space and $T: V \rightarrow V$ a function (or transformation) from $V$ to $V$. What does it mean to say that $T$ is linear? (There are two conditions that must be satisfied.) If $T$ is linear what is $T(0)$ ?
(16) What does it mean to say that a subset $W \subset \mathbb{R}^{n}$ is a subspace? (There are 3 conditions that must be satisfied.) If $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a linear transformation and $\lambda \in \mathbb{R}$, let $W$ be the subset of $\mathbb{R}^{n}$ consisting of all the vectors $\mathbf{v}$ such that $T(\mathbf{v})=\lambda \mathbf{v}$. Show that $W$ is a subspace of $\mathbb{R}^{n}$. [Remark: The subspace $W$ is the eigenspace $E_{\lambda}$ consisting of all the eigenvectors of $T$ with eigenvalue $\lambda$ together with the zero vector.]
(17) What is the rank-nullity theorem? If $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ is a linear transformation, what can you say about the kernel of $T$ if $n>m$ ?
(18) Let $V$ be a linear space and $\mathcal{B}$ a basis of $V$. Let $T: V \rightarrow V$ be a linear transformation. What is the $\mathcal{B}$-matrix of $T$ and how can it be computed? In each of the following examples, write down a basis $\mathcal{B}$ of $V$, compute the $\mathcal{B}$-matrix of $T$, and determine whether $T$ is an isomorphism.
(a) $V=\mathcal{P}_{2}$, the linear space of polynomials $f(x)$ of degree $\leq 2$, and $T: V \rightarrow V, T(f(x))=f(x)+f^{\prime}(x)+f^{\prime \prime}(x)$.
(b) $V=\mathbb{R}^{2 \times 2}$, the linear space of $2 \times 2$ matrices, and $T: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$, $T(X)=A X+X B$ where $A=\left(\begin{array}{ll}1 & 0 \\ 2 & 3\end{array}\right)$ and $B=\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$.

