

(1)

M235 COMMON EXAM PRACT PROBLEM ANSWERS

(1) v is a lin comb of $v_1, \dots, v_m \Leftrightarrow v = \sum_{i=1}^m a_i v_i \quad a_i \in \mathbb{R}$

(2) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \Leftrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{array} \right)$ corresponds to eq. with solutions

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 5 \\ 0 & 1 & 2 \end{array} \right) \leftarrow \text{oops no solution}$$

It is not a lin comb.

(3). Span of v_1, \dots, v_m is set $\left\{ \sum_{i=1}^m a_i v_i \mid a_i \in \mathbb{R} \right\}$ or set of all lin comb of v_1, \dots, v_m .

(4) (a) $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Span is all of \mathbb{R}^2 .

(b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. Span is line thru $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$\text{since } \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

(c) This is a plane in \mathbb{R}^3 .

(d) $\left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \left(\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right), \left(\begin{array}{c} 2 \\ 3 \\ 5 \end{array} \right)$ spans all of \mathbb{R}^3 .

$$\text{Why? } \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 2 & 3 & b \\ 1 & 3 & 5 & c \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & b-a \\ 0 & 2 & 3 & c-a \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 1 & (c-a)-2(b-a) \end{array} \right)$$

↓ now
obvious that
 $\text{rank} = 3$.

(e) $\left(\begin{array}{c} 1 \\ -1 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right), \left(\begin{array}{c} 2 \\ 4 \\ -6 \end{array} \right)$? Does this span all of \mathbb{R}^3

$$\left(\begin{array}{ccc} 1 & 0 & 2 \\ -1 & 1 & 4 \\ 0 & -1 & -6 \end{array} \right) \rightarrow \left(\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & -1 & -6 \end{array} \right) \rightarrow \left(\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{array} \right) \rightarrow \text{rank} = 2$$

so spans a plane.

235 - Answers

(2)

⑤ $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$. (a) T is a lin trans $\Leftrightarrow T(v+w) = T(v)+T(w)$
 $T(\lambda v) = \lambda T(v)$

$$\lambda \in \mathbb{R}, v, w \in \mathbb{R}$$

(b). Standard matrix of T = ?

(c). n - columns, m rows

(d). I think that he means * matrix of T WRT standard basis. So columns of A are $T\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ expressed in terms of standard basis.

⑥ (a) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+1 \\ z \\ x \end{pmatrix}$ is not a linear map since

$$T\begin{pmatrix} xy \\ zx \\ yx \end{pmatrix} = \begin{pmatrix} xy+1 \\ zx \\ yx \end{pmatrix} \neq T\begin{pmatrix} y+1 \\ z \\ x \end{pmatrix} = \begin{pmatrix} z+1 \\ x \\ y \end{pmatrix}.$$

(b) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ x+y+z \end{pmatrix}$ is linear. We \checkmark one (but not other)
 property

$$T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \stackrel{?}{=} T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$$

$$(x_1+x_2)+2(y_1+y_2)+3(z_1+z_2)$$

$$(x_1+x_2)+y_1+y_2+z_1+z_2$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1+2y_1+3z_1 \\ x_1+y_1+z_1 \end{pmatrix} + \begin{pmatrix} x_2+2y_2+3z_2 \\ x_2+y_2+z_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1+x_2+2(y_1+y_2)+3(z_1+z_2) \\ x_1+x_2+y_1+y_2+z_1+z_2 \end{pmatrix}$$

? - Yes.

Answers(6c). Rotation by $\pi/4$ is linear. Its matrix

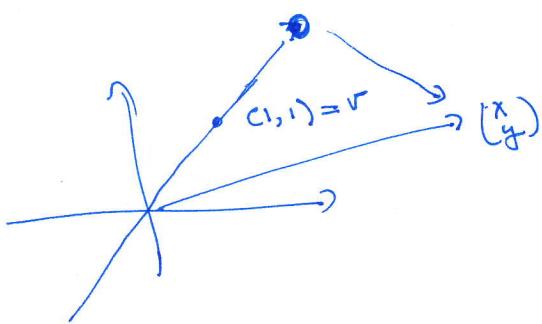
$$\text{is } \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}$$

(3)

(6d)

T is linear. Its matrix is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(6e)



$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} + w \quad \Rightarrow \quad \bullet \text{ with } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let P denote proj.
Want a formula for $P(x)$

$$x+y = x \cdot 2 + 0$$

$$\frac{x+y}{2} = \lambda.$$

$$P\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x+y}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \text{let } \begin{matrix} x=1 \\ y=0 \end{matrix} \text{ get } P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\therefore P \leftarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$$P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}. \quad \checkmark$$

$$\checkmark: P^2 = P \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$(f)_{(a)} \ker T = \{x \mid Tx = 0\} \subseteq \mathbb{R}^n$$

$$\text{im } T = \{y \in \mathbb{R}^m \mid \exists x \text{ with } y = Tx\}.$$

(b) image is span of col vectors

(c) ker - use row reduction.

(4)

Math 235 Review

$$9. (a) T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}. \quad \begin{array}{l} x = -2y \\ y = y \end{array}$$

\therefore ker is span of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Image is span of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

But $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is redundant. So just the span of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(b) Image is plane $x+2y+3z=0$

Kernel is set of vectors \perp to $x+2y+3z=0$. It is the kernel of the matrix $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. Use row reduction on this.

We get $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

$$\begin{array}{l} x = -2y - 3z \\ y = y \\ z = z \end{array} \quad \text{no set of solutions is spanned by } \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \quad \text{kernel is spanned by}$$

9c.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{array}{l} x = z \\ y = -2z \\ z = z \end{array} \quad \text{using } z$$

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ spans kernel.

The image is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

(5)

23.5 Answers

$$\{v_1, \dots, v_m\}$$

(10). linearly ind $\Leftrightarrow \left(\sum a_i v_i = 0, a_i \in \mathbb{R} \right) \Rightarrow \forall a_i = 0$

If lin ind, $m \leq n$.

(11). a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. yes ind. Why? $x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$11b. \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \begin{pmatrix} 12 \\ 37 \end{pmatrix}$$

can't be lin ind. since there are 3 vectors in \mathbb{R}^2 .

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \text{now}$$

compute kernel of $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \rightarrow \mathbb{R}$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

no free variables \Rightarrow
 $\text{ker} = \{0\}$.

(11c) $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} \begin{pmatrix} 0 \\ 13 \\ 7 \end{pmatrix} \rightarrow \text{look at ker } \begin{pmatrix} 0 & 2 & 0 \\ 3 & 5 & 13 \\ 0 & 0 & 7 \end{pmatrix}$. Use row reduction

to find if $b = 0$ or ~~ker $\neq 0$~~ .

$$\begin{pmatrix} 0 & 2 & 0 \\ 3 & 5 & 13 \\ 0 & 0 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 5 & 13 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix} \rightarrow \text{well} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \text{Inspection} \therefore \text{ker} \neq 0,$$

\therefore vectors are ind.

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot x + 4 \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 13 \\ 0 \\ 7 \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{cases} 7z = 0 \Rightarrow z = 0 \\ 2y = 0 \Rightarrow y = 0 \end{cases} \Rightarrow 3x = 0 \Rightarrow x = 0 \text{ also.}$$

11d. $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 5 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{free variable}$

$\Rightarrow \text{ker} \neq 0$.

\Rightarrow (dependent \Rightarrow not ind.).

235 Answers.

(6)

- (12) W is a subspace $\Leftrightarrow w \in W, u \in W \Rightarrow u + w \in W$
 and
 $w \in W, \lambda \in \mathbb{R} \Rightarrow \lambda w \in W$.
 $0 \in W$,

- (13) (a) no. $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin W$.

(b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + 2y + 3z = 0 \right\}$

This is a subspace.
 Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be
 given by matrix $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.
 Then $\ker T = W$ but the
 ker of lin map is subspace

- (c). ?.

(d). Both are subspaces

(e). Yes it is a kernel of linear map.

- (e). Yes subspace

(f). Basis \Leftrightarrow (1) spans W and (2) independent

(g) Basis \Leftrightarrow (1) spans W and (2) independent

(h) $\dim W = \# \text{ of elts in basis of } W$

(i) Row reduction

(15) (a) $x + 2y + 4z = 0 \rightarrow \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$

$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis

$$\begin{cases} x = -2y - 2z \\ y = y \\ z = z \end{cases} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Answers $x_4 = twu$

15b. $\begin{pmatrix} 10 & 305 \\ 01 & 207 \\ 00 & 012 \end{pmatrix}$

$$\begin{aligned} x &= -3z - 5u \\ y &= -2z - 7u \\ z &= z \\ w &= -2u \\ u &= u \end{aligned}$$

(7)

$$\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -5 \\ -7 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

basis of kernel

15c. $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 4 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Basis of the image is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

15d. $W = \mathbb{R}^n \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$. is a basis.