

①

Math 235 Review Common Midterm Answers II

17. $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$. $\text{Rank}_f T = \# \text{ of leading ones in RREF of matrix of } T$
 $= \dim \text{ of the image of } T$

$$n = \dim(\text{image of } T) + \dim(\ker T) = \text{rank} + \text{nullity}$$

18. a) $\{v_1, \dots, v_m\}$ is lin ind $\Rightarrow m \leq n$. $\{v_i \in \mathbb{R}^n\}$

(b) $\{v_1, \dots, v_m\}$ spans $\mathbb{R}^n \Rightarrow m \geq n$.

(c) $\{v_1, \dots, v_m\}$ is a basis $\Rightarrow m = n$.

$$(18) \quad \begin{array}{l} \text{(a) } T: \mathbb{R}^4 \rightarrow \mathbb{R}^3 \quad \begin{matrix} \dim \text{im } T \leq 3 \\ \dim \ker T \geq 5 \end{matrix} \\ \quad \quad \quad \text{also } \dim \ker \leq 8 \end{array} \quad \left| \begin{array}{l} \text{(b) } T: \mathbb{R}^4 \rightarrow \mathbb{R}^7 \\ 0 \leq \dim \text{image of } T \leq 4 \\ 0 \leq \dim \ker T \leq 34 \end{array} \right.$$

c) $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$.
if $n \geq m \rightarrow \begin{matrix} \dim \ker \geq n-m, \dim \text{image} \leq n \\ 0 \leq \dim \ker \leq n, \dim \text{image} \leq n. \end{matrix}$

19. a) $\dim W \leq n$. b) $v = \sum a_i v_i \quad \{v_i\} = \text{Basis}$
 $[v]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$

20. a) Basis - v1, v2 are ind.
(2) $\dim W = 2$ [use rank nullity with the map $\mathbb{R}^3 \rightarrow \mathbb{R}$
 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow x+y+z$]

$\therefore \{v_1, v_2\}$ is a basis

$$(b) \quad [w]_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Rightarrow w = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

$$(c) \quad v = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} \quad r \in W \text{ because } 3+4-7=0$$

$$\begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} 3 = a_1 + 0 \\ 4 = a_1 + a_2 \\ -7 = -a_2 \end{cases} \quad \begin{cases} a_1 = 3 \\ a_2 = 7 \end{cases}$$

Math 235 Review Answers

1) (2)

21: a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis because ind and spans. - Subst one implies the other.

$$\text{Other: } x \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x+2y=0 \\ 2x+y=0 \end{array} \rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \downarrow \begin{pmatrix} 0 & 1 \end{pmatrix}$$

(b). $L^{-1}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$L^{-1} \cdot \begin{bmatrix} a \\ b \end{bmatrix}_B = av_1 + bv_2.$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = L^{-1}.$$

(22). $T_B \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow T \left\{ \sum a_i v_i \right\} = \sum c_i v_i$

$$(b) T_B \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} T_B \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_B = T(v_i) = \sum c_i v_i \text{ gives 1st col - others are done in the same way.}$$

(c). Let $I_{B \leftarrow E}$, $I_{E \leftarrow B}$ be coordinate matrices

$$I_{E \leftarrow B} T_B I_{B \leftarrow E} = T_E.$$

(23). $A_E = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad I_{E \leftarrow B} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

$$A_B = I_{B \leftarrow E} A_E I_{E \leftarrow B} \quad I_{B \leftarrow E} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(24) (a). $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$(b) (2x+y+z) \rightarrow (2 \ 1 \ 1) \rightarrow (1 \ y_2 \ y_2) \rightarrow \begin{cases} x = -y_2 y - y_2 z \\ y = y \\ z = z \end{cases}$$

$$f_1 = \begin{pmatrix} -y_2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -y_2 \\ 0 \\ 1 \end{pmatrix} = f_2, \quad f_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(24c) \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = f_1 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} = f_2 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f_3 \quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{IJ. } ③$$

25. (a) $P_4 \hookrightarrow \{1, x, x^2, x^3, x^4\}$.

$$(b) \quad \mathbb{R}^{2x3} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(c) \quad P_n \rightarrow h x, x^2, \dots, x^n$$

(d) See (b).

27. (a) $V = P_2$, $W = \{f \mid f(1) = 0\}$ is a subspace of $\{x(x-1), x(x-1)^2\}$.

$$(b) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} a+b & a-b \\ c+d & c-d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a-c & b-d \end{pmatrix} \Leftrightarrow \begin{array}{l} a+b = a+c \\ a-b = b+d \\ c+d = a-c \\ c-d = b-d \end{array}$$

These are all linear equations:

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & -2 & 0 & -2 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

now find kernel.

$$\begin{array}{l} b-c=0 \\ a-2b-d=0 \\ -a+2c+d=0 \\ c-b=0 \end{array}$$

$$(c) \quad -(a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0) = a_4 x^4 - a_3 x^3 + a_2 x^2 - a_1 x + a_0$$

\Downarrow

$-f(x)$

$$\Rightarrow a_0 = a_2 = a_4 = 0, \quad \text{so } \{x^3, 1\}.$$

Math 235 Answers .

I - (4)

(28) $T: P_3 \rightarrow P_3$ $T: f \rightarrow f'$ is linear $\ker T = \text{span of } 1$.
 $\text{im } T = \text{poly of degree} \leq 2$

(b) T is linear. $\ker T = \text{space spanned by } (x-1)(x-2)$.
 $\text{image of } T \text{ is all of } \mathbb{R}^2$.

(c) T is linear. $\ker T = \text{span of the poly } x$.

$$\begin{aligned} 1 &\rightarrow -1 \\ x &\rightarrow 0 \\ x^2 &\rightarrow 2x^2 - x^2 = x^2 \end{aligned} \quad \therefore \text{image is spanned by } 1 \text{ and } x^2.$$

(29) Assume $T: V \rightarrow W$ is linear and $\dim V = \dim W$.

(a) $\ker T = 0$, T is invertible (b) If $\text{im } T = \text{all of } W$, T is invertible

(30) $T: P_3 \rightarrow \mathbb{R}^2$ (a) $T: P_3 \rightarrow \mathbb{R}^2$ is not iso, $\dim P_3 = 4$
 $\dim \mathbb{R}^2 = 2$.

(b) $T: P_2 \rightarrow P_2$ is ~~not~~ an iso. $Tf = f - f'$ never = 0 since
 $\deg f > \deg f'$ so $f \neq f'$ any polynomial.

(c) $\det A = -2 \therefore A$ is invertible $\det B = -2 \therefore B$ is
invertible. $\exists Y = A^{-1}B \Rightarrow A^{-1}YB^{-1} = X$. This
exhibits the inverse.

Math 235 Common Review Answers

#-5

#31 $B = \{f_1, \dots, f_n\}$ is a basis of V . Let $v \in V$.

Then $[v]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ means $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$.

#32. $f = 1x^2 + 2x + 3 = a(x-1)^2 + b(x-1) + c \quad a, b, c = ?$

$$= ax^2 - 2ax + a + bx - b + c$$

$$= ax^2 + [b-2a]x + [a-b+c]$$

$$\left. \begin{array}{l} a=1 \\ b-2a=2 \\ a-b+c=3 \end{array} \right\}$$

$$\text{need to solve. } a=1 \Rightarrow b-2=2 \Rightarrow b=4.$$

$$a-b+c=3 \rightarrow 1-4+c=3 \Rightarrow c=6$$

$$[f]_B = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \text{ with respect to } B = \{(x-1)^2, (x-1), 1\}$$

33. $T: V \rightarrow V$, B = basis. How to compute columns of T
 $= \{f_1, \dots, f_n\}$ with respect to basis B .

Answers. ① $\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}_B \rightarrow 1 \cdot f_1 + 0 \cdot f_2 + \dots + 0 \cdot f_n = f_1 \xrightarrow{T} * \in V$.

now write $* = a_1 v_1 + \dots + a_n v_n$. Then

$$T: \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}_B \rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_B \quad \therefore 1^{\text{st}} \text{ col is } \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \text{ Do this}$$

with $\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ etc.

34. $\text{Let } T: f \rightarrow f' \quad f \in P_3$. Write matrix for f wrt $\{1, x, x^2, x^3\} = B$
 $= \text{basis of } P_3$

$$\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{1} T \xrightarrow{0} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \text{ Continued on next}$$

Math 235 - Review Common Answers

#-6

34a. $T: f \rightarrow f'$ $P_3 \rightarrow P_3$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_B \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_B \Rightarrow 1^{\text{st}} \text{ col is } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_B \longleftrightarrow f = x \xrightarrow{T} 1 \longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_B \Rightarrow 2^{\text{nd}} \text{ col is } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_B \longleftrightarrow f = x^2 \rightarrow 2x \longleftrightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 3^{\text{rd}} \text{ col is } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_B \longleftrightarrow f = x^3 \rightarrow 3x^2 \longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} \Rightarrow 4^{\text{th}} \text{ col is } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

34b. $T: P_2 \rightarrow \mathbb{R}^2$ $f \rightarrow xf' - f$ $(34b \text{ doesn't really apply})$

Basis $\{1, x, x^2\} = C$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_C \longleftrightarrow 1 \rightarrow x \cdot 0 - 1 = -1 \longleftrightarrow \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}_C$
 $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_C \longleftrightarrow x \rightarrow x \cdot -x = 0 \longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_C \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_C$
 $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_C \longleftrightarrow x^2 \rightarrow 2x^2 - x^2 = x^2 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_C$

ZERO FUNCTION

35. (a) $\{ \cos x, \sin x \}$ spans by definition of V
 $\{ \cos x, \sin x \}$ is independent: $a \cos x + b \sin x = 0$

$$\Rightarrow \text{at } x=0 \text{ get } a \cos 0 + b \sin 0 = 0 \Rightarrow a \cdot 1 = 0 \Rightarrow a = 0$$

$$\Rightarrow \text{at } x=\pi/2 \text{ get } a \cos \pi/2 + b \sin \pi/2 = 0 \Rightarrow b \cdot 1 = 0 \Rightarrow b = 0$$

(b) $T: V \rightarrow V$, $f \rightarrow f' + f$. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_B \longleftrightarrow \cos x \rightarrow \begin{pmatrix} \cancel{-\sin x} \\ \cancel{\sin x + \cos x} \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}_B$

$$B = \{ \cos x, \sin x \}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_B \longleftrightarrow \sin x \rightarrow \cos x + \sin x \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}_B$$

∴ matrix is $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}_B$

T is invertible since
det is $\neq 0$

Math 235 Review Answers

II 7

37. $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid (1-z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$ $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis
of V . So is $C = \{w_1, w_2\} = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(a) Compute Change of Basis Matrix

$$S_{B \rightarrow C} = \sum_{v \in B} v \text{ takes } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B \rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_C$$

$$(a1) \text{ mean } v = 1 \cdot v_1 + 0 \cdot v_2 = a_1 w_1 + a_2 w_2$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \Leftrightarrow \begin{matrix} 1 = 2a_1 + a_2 \\ 0 = -a_1 - a_2 \\ 0 = a_2 \end{matrix} \Rightarrow \begin{matrix} a_2 = 1 \\ a_1 = -1 \end{matrix}$$

so $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is 1st col of $S_{B \rightarrow C}$

$$(a2). \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_B \quad b_1 w_1 + b_2 w_2 = b_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow v = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \Rightarrow \text{equations} \quad \begin{matrix} 0 = b_1 \\ 1 = b_2 \\ -2 = -1 \cdot b_1 + -2 \cdot b_2 \end{matrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \text{ is second col.}$$

$$S_{B \rightarrow C} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

$$(b) S_{C \rightarrow B} = (S_{B \rightarrow C})^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{-2}.$$



$$\det T = 4 - 6 = -2. \quad \therefore \text{area is } |-2| = 2.$$

T mult area by 2 so area of \bar{K} is $2 \cdot \pi$.