

①

① Math 235
Practice I

$$\begin{array}{l} 1. \quad x_1 + 2x_2 - x_4 = 1 \\ \quad x_3 + 2x_4 = 1 \\ \quad 2x_1 + 4x_2 - 3x_3 - 8x_4 = -1 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 4 & -3 & -8 & -1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -6 & -3 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Circled entries are pivots
or leading ones.
This is in RREF.

The "free variables" are x_2, x_4 .

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \leftarrow \begin{cases} x_1 = 1 - 2x_2 + x_4 \\ x_2 = x_2 \\ x_3 = 1 \\ x_4 = x_4 \end{cases}$$

$$2. \quad A = \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ -2 & 1 \end{pmatrix} \quad \text{Q: AB is not defined}$$

$$\textcircled{3} \quad BA = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 11 & -2 \\ 1 & -b \end{pmatrix}. \quad \text{2b: B gives a map from } \mathbb{R}^2 \text{ to } \mathbb{R}^3.$$

$$3. \quad u = \lambda v + y \quad (\lambda v = x) \quad u = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

~~dot with v~~

$$\langle u, v \rangle = \langle \lambda v, v \rangle + \langle y, v \rangle \rightarrow \frac{\langle u, v \rangle}{\langle v, v \rangle} = \lambda = \frac{-1+1}{1+1+1} = \frac{1}{3}$$

~~0 smu y1v~~

$$\therefore x = \lambda v = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad y = u - \lambda v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -4/3 \\ 4/3 \end{pmatrix}.$$

4a: $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \quad \text{Find } A^{-1}$

$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} R_2 - 1.5R_1 \\ R_1 \leftrightarrow R_2 \end{pmatrix}} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & -1/2 & 1/2 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} R_2 \cdot (-2) \\ R_1 + R_2 \end{pmatrix}} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & -2 \end{pmatrix} \xrightarrow{\begin{pmatrix} R_1 \cdot 1/2 \\ R_2 \cdot (-1) \end{pmatrix}} \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 2 \end{pmatrix}$

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(2) (2)

(4) (a) $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Find A^{-1} .

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{2} & 1 \end{array} \right)$$

Inverse

(b) $\begin{cases} 2x+ty=1 \\ 3x+2y=2 \end{cases} \rightarrow \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \text{multiply by } A^{-1}. \\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad x=0, y=1$

(5) $\text{ker } F = \{ v \in \mathbb{R}^m \mid F(v) = 0 \}$. $S \subseteq \mathbb{R}^m$ is a subspace if (i) $u, v \in S \Rightarrow u+v \in S$ and (ii) $u \in S, \lambda \in \mathbb{R} \Rightarrow \lambda u \in S$

(b) $x = \sum a_i u_i$, $u_i \in S$ says x is a linear combination of elems of S

(c). The rank of a matrix is the number of leading ones you get after using row operations to put the matrix in RREF.

(6) $\begin{array}{c|ccccc} & & 1 & 2 & 3 & 4 \\ \uparrow & & \downarrow & & \downarrow & \\ M & & 1 & 2 & 3 & 4 \end{array}$ M has rank 2. There are 3 variables
Thus there is 1 free variable. False.

(7) Rank of M is four. If it were less than 4 there would be free variables.

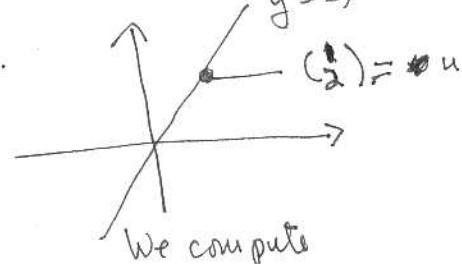
~~rows of all zeros.~~

Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be an b vector. Write

$$v = au + w \text{ with } w \perp u$$

$$a = \frac{\langle u, v \rangle}{\langle u, u \rangle} = \frac{\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle}{\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle} = \frac{x+2y}{5} \quad (*)$$

au is Proj to our line. Call projection P.



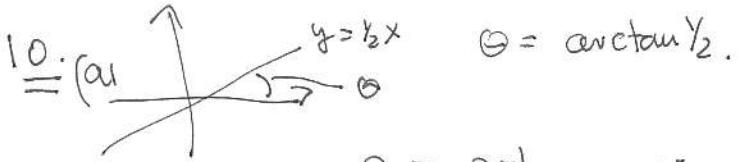
We compute

$$u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = au = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix} \quad (\stackrel{a}{=} \text{from equation } (*))$$

$$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = au = \frac{2}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 4/5 \end{pmatrix} \quad \cdot P = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}$$

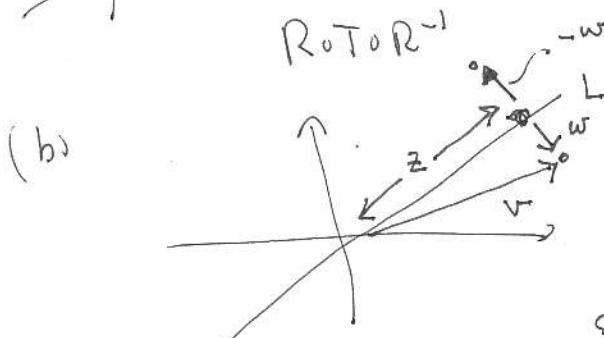
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9. $M = \begin{pmatrix} 7 & 2 \\ 3 & 8 \end{pmatrix}$ $M \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \begin{pmatrix} 80 \\ 70 \end{pmatrix}$. 3



$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



The vector $(2, 1) = u$ is on L .

projection of v is denoted by

$$\text{so } v = z + w.$$

The reflection of v is $z - w$.

$$v = au + w = z + w$$

$$a = \frac{\langle u, v \rangle}{\langle u, u \rangle} = \frac{2x+y}{5}$$

$$\text{so } z = \left(\frac{2x+y}{5} \right) \cdot u$$

$$w = v - z = \left(\begin{matrix} x \\ y \end{matrix} \right) - \left(\frac{2x+y}{5} \right) \left(\begin{matrix} x \\ y \end{matrix} \right)$$

We compute for any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$. ^{step ①} we get $v = au + w = z + w$.
^{step ②} Set $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $z = \left(\frac{2}{5} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix}$

$$w = v - z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}$$

Reflection of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $z - w = \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} - \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$

Set $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $z = \left(\frac{1}{5} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix}$ ~~$z = \left(\frac{1}{5} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$~~ : $\Rightarrow z - w = \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix} - \begin{pmatrix} -2/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$

$$\therefore \text{matrix is } \begin{pmatrix} 4/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$$

11. (a) $F: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear $\Leftrightarrow F(u+v) = F(u)+F(v)$ and $F(\lambda u) = \lambda F(u)$ with $u, v \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$.

(b1) $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} xy-1 \\ 3x-y \end{pmatrix}$ is not linear. Why? If F is linear $F(g) = F(0 \cdot g) = 0 \cdot F(g) = 0$.
 and that is not case $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

(b2) $x \rightarrow \sin x$ is not linear $F(x,y) = \sin(x+y) + \sin x + \sin y$

$$\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, v_1 + v_2 \rangle = \left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, v_1 \right\rangle + \left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, v_2 \right\rangle$$

$$\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \lambda v \rangle = \lambda \langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, v \rangle$$

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(4)

12. Define linear combination. Let $\{u_1, \dots, u_n\} \subseteq \mathbb{R}^n$, a linear comb of u_1, \dots, u_n is any element of the form $\sum_{i=1}^n d_i u_i$ $d_i \in \mathbb{R}$.

Let U be a subspace of \mathbb{R}^n . The set $S \subseteq \mathbb{R}^n$ spans U if every element in

U can be written as a linear combination of elements of S .

A subset $S \subseteq \mathbb{R}^n$ is a subspace provided $u, v \in S \Rightarrow u+v \in S$ and $c \in \mathbb{R}, u \in S \Rightarrow cu \in S$

Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear. $\ker F = \{x \in \mathbb{R}^n \mid F(x) = 0\}$

The nugage of $F = \{b \in \mathbb{R}^m\}$ there exists $x \in \mathbb{R}^n$ so that $Fx = b\}$

The rank of a matrix is the number of leading ones when the matrix has been put in RREF using row operations.

Now $b = 0$ gives

13. (a) True. Invertible \Rightarrow one-to-one $\Leftrightarrow Fx = b$ has a ! solution.

state ment

(b) True. Subspace \Rightarrow closed under + and scalar mult.

(c) Yes. True. Rank 5 \Rightarrow $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ in RREF - no row of all zeros.

(d) False. $B\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow B\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(e) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow$ rank mat 3 \Rightarrow at least two cols with no leading ones \Rightarrow 2 variables free to assign.

$$(14) \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 2 & | & 1 \\ 0 & 2 & 0 & | & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 2 & | & 1 \\ 0 & 2 & 0 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

$$x = 1, y = 0, z = 0. \text{ We check } \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \checkmark$$

(15)

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$$\left(\begin{array}{ccccc} 1 & -1 & 2 & 3 & 1 \\ -1 & 1 & -1 & -4 & 1 \\ 1 & -1 & 3 & 2 & 3 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Kernel: ~~$x_1 = x_2 = x_3 = 0$~~

kernel: $\left\{ \begin{array}{l} x_1 = x_2 = -5x_4 + 3x_5 \\ x_2 = x_2 \\ x_3 = x_4 - 2x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{array} \right.$

so $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

spans the kernel and has no redundant elts.

$$\left(\begin{array}{ccccc} 1 & -1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Free variables x_2, x_4, x_5

$$\left\{ \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right\} = A \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \left(\begin{pmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + u \left(\begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right) \right) \right)$$

image. The columns of our matrix span the image:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 3 \end{pmatrix} \right\}$$

More: From $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$ we get $A \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$.

(a) But $A \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) = 1 \cdot A_1 + 1 \cdot A_2 = 0 \Rightarrow A_2 = -A_1$, so A_2 is redundant.

$$A \left(\begin{pmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \Rightarrow -5A_1 + A_3 + A_4 = 0 \Rightarrow A_4 = 5A_1 - A_3 \Rightarrow A_3 \text{ is redundant}$$

(b) ~~$A \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$~~

$$A \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \Rightarrow -5A_1 + A_3 + A_4 = 0 \Rightarrow A_4 = 5A_1 - A_3 \Rightarrow A_3 \text{ is redundant}$$

(c). $A \left(\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \Rightarrow A_5 = 2A_3 - 3A_1 \Rightarrow A_5 \text{ is redundant.}$