Math 235 Practice Problems for Exam October 4, 2013

Name:

Show All Work

1: Find all the solutions, if any, to the system of linear equations

$$\begin{aligned} &x_1 + 2x_2 - x_4 = 1 \\ &x_3 + 2x_4 = 1 \\ &2x_1 + 4x_2 - 3x_3 - 8x_4 = -1. \end{aligned}$$

Use Gauss elimination.

2: Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ -2 & 1 \end{pmatrix}.$$

a: Compute AB if possible. Compute BA if possible.

b: For appropriate choice of a, b the matrix B gives a function with domain $\mathbb{R}^a$ and target $\mathbb{R}^b$. What are the numbers a and b?

3: Let $u = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Write $u = x + y$ with x a scalar multiple of v and y orthogonal to v. This means that you are to find explicitly the two vectors x, y.

4a: Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Compute A^{-1}. You must show your work.

4b: Use your result in problem 4a to solve the system of equations

$$\begin{aligned} 2x + y &= 1 \\ 3x + 2y &= 2 \end{aligned}.$$

5a: Let $F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be linear. Define $ker(F)$. Define what it means to be a subspace of $\mathbb{R}^m$.

5b: Let S be a subset of $\mathbb{R}^n$. What does the phrase "x is a linear combination of the elements of S" mean? What does it mean to say that the span of S is all of $\mathbb{R}^n$.

5c: What is the rank of a matrix?

6: True or False. Justify your answer. Let M be a matrix of size 4×3. Assume that $Mx = b, b \in \mathbb{R}^4$ has a solution and the rank of M is 2. Then there are 3 free variables.

7: Let M be a matrix of size 4×4 and $b \in \mathbb{R}^4$. We are given that the equation $Mx = b$ has a solution and it is unique. What is the rank of M? Justify your answer.

8: Find the matrix of orthogonal projection onto the line $y = 2x$.

9: A town has two supermarkets, one named W and one named S. Each month 70% of the customers of W stick with W and 30% switch to S. Each month 80% of the customers of S stick with S and 20% switch to W. The total number of coustomers stays constant from month to month. Let $w(n), s(n)$ denote the number of customers of W, S during month n. Find a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that

$$\begin{pmatrix} w(n+1) \\ s(n+1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w(n) \\ s(n) \end{pmatrix}.$$

Assume $w(0) = 100, s(1) = 50$. Compute, using M, the number of customers in month 1 of S, W, that is, compute $w(1), s(1)$.

10: Find the matrix M which represents reflection about the line L given by the equation

$$y = (1/2)x$$

in two ways:

10a: By writing the composition as a composition of rotations and reflection about the x-axis. Note that the line L makes an angle $\pi/6$ with the x-axis.

10b: By using projection onto the line L to compute $M\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $M\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

11 a: Define what it means for a function $F : \mathbb{R}^m \to \mathbb{R}^n$ to be a linear transformation.
b: Are the following linear transformations? Why? Note that the why part of the question is very important.
b1: $F : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y - 1 \\ 3x - y \end{pmatrix}$
b2: $F : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$

b3: $F : \mathbb{R}^3 \to \mathbb{R}, \vec{v} \mapsto (-1, 2, 3) \cdot v$.

12: Define the following terms:
a: linear combination
b: spans a subspace
c: subspace
d: kernel
e: image
f: rank.
You should say what kind of object each term applies to. For example we say "the rank *of a matrix*" or "*of a linear map*".

13: True or False. You must explain the reason for your answer.

a: If a matrix M is invertible, then kernel of M is just the zero vector.

b: If $u, v, w \in V, V$ a subspace of $\mathbb{R}^n$, then the vector $2u - 3v + 4w$ is also an element of V.

c: If a 5 by 5 matrix A has rank 5, then any linear system of equations with coefficient matrix A will have a unique solution.

d: There exists a 2×2 matrix B so that

$$B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and

$$B \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

e: Let A be a matrix of size 3×5. Assume that the equation

$$Ax = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

has a solution. Then it has infinitely many solutions.

f: Let M be a matrix of size 5×5. Assume that for some $b \in \mathbb{R}^5$ the equation $Ax = b$ in unknown x has a unique solution. Then A is invertible.

14: Let

$$a = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \ c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Express

$$d = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

as a linear combination of a, b, c.

15: Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 & 1 \\ -1 & 1 & -1 & -4 & 1 \\ 1 & -1 & 3 & 2 & 3 \end{pmatrix}.$$

15a: Find a set of vectors that spans the image of A.

15b: Find a set of vectors that spans the kernel of A and has no redundant elements.