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On Schrödinger and Wave Maps

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Abstract. We survey recent work, joint with Atanas Stefanov and Karen Uhlenbeck, on Schrödinger maps and Wave maps from Minkowski space $\mathbb{R}^{n+1}$ into (compact) Riemaniann manifolds. We focus on the results obtained and some of the methods from harmonic analysis and gauge theory used.

Introduction

This expository article arose from a lecture given at the Summer Research Conference on Harmonic Analysis held at Mt. Holyoke, Massachusetts in the summer of 2001. The results we present are joint work with Atanas Stefanov and Karen Uhlenbeck and appear in [**33**] [**34**].

There are three evolution equations which are derived from the same geometric considerations. The best known one is the heat flow for harmonic maps. This flow equation has been successfully studied by methods which in spirit depend on the same geometric ideas used in the elliptic theory of harmonic mappings.

The second equation, the Schrödinger map equation, has a number of different descriptions which are all equivalent for smooth solutions. For maps onto the -sphere $\mathbb{S}^2$, for example, a simple description can be given in terms of the stereographic projection of the sphere onto the complex plane. The energy of the map $s : \mathbb{R}^n \to \mathbb{C} \cup \infty$ is

$$E(s) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^{n} \frac{|\partial s/\partial x_j|^2}{(1+|s|^2)^2} \, (dx)^n.$$

A simple calculation then shows that the Euler-Lagrange equations of the equations for a *harmonic map* can be written as

$$\sum_j \nabla_j \frac{\partial s}{\partial x_j} = 0 \qquad \text{where} \qquad \nabla_j = \frac{\partial}{\partial x_j} - 2(\frac{\partial s}{\partial x_j}\, \overline{s})/(1+|s|^2)$$

is the covariant derivative (corresponding to the pull-back of the Levi-Civita connection on the tangent plane to the sphere by the map s). The *Schrödinger map*

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equation is

$$\frac{\partial s}{\partial t} = \pm i \sum_j \nabla_j \frac{\partial s}{\partial x_j}. \tag{S}$$

This is a nonlinear Schrödinger equation with a non-polynomial nonlinearity including derivatives. Note the heat equation would be given by (S) without the $\pm i$ on the right hand side.

From a physical viewpoint equation (S) arises also in a natural fashion from the Landau-Lifschitz equation for a macroscopic ferromagnetic continuum for $s : \mathbb{R}^n \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and

$$\frac{ds}{dt} = -s \times \Delta s.$$

This is the Schrödinger part of the mixed parabolic-Schr̈odinger Ginzburg-Landau-Lifchitz model describing the evolution of magnetization in a material [**32**], [**44**] [**35**].

The third equation is the wave map equation between two Riemannian manifolds, which has extensively been studied by a number of mathematicians in the last decade. It is the natural Minkowskian analogue of the harmonic map equation. For a map $s : \mathbb{R} \times \mathbb{R}^n \to M$, where $(M, \mathfrak{g})$ is -for example- a compact Riemaniann manifold, the critical points of the Lagrangian

$$\int\int |\frac{\partial s}{\partial t}|^2_{\mathfrak{g}} - |\nabla_x s|^2_{\mathfrak{g}} dx dt$$

are the solutions to the *wave map equation.* This equation can be regarded as given through covariant derivatives and in coordinate free notation is given by:

$$s^*\nabla_0 \frac{\partial s}{\partial t} - \sum_{j=1}^{n} s^*\nabla_j \frac{\partial s}{\partial x^j} \;=\; 0. \tag{W}$$

where $s^*\nabla$ are the covariant derivatives (corresponding to the pullback of the Levi-Civita connection on M via the map s). This is a nonlinear wave equation with a non-polynomial nonlinearity including derivatives.For $M = \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ for example, equation (W) looks like

$$\Box s = -\frac{\partial^2 s}{\partial t^2} + \Delta s = -s\,(|\nabla_x s|^2 - |\frac{\partial s}{\partial t}|^2).$$

The wave equation models the propagation of different kind of waves -such as light waves- in homogeneous media. Nonlinear models of conservative type arise in quantum mechanics while other variants appear for example in the study of vibrating systems and semiconductors. The nonlinear Schrödinger equation arises in various physical contexts in the description of nonlinear waves- such as propagation of a laser beam in a medium whose index of refraction is sensitive to the wave amplitude. Some of the interesting questions are those about local and global existence of solutions, uniqueness as well as long time behavior of global solutions.

To illustrate the general framework in which these problems are classically treated let us consider for a moment a semilinear Schrödinger equation. The existence of a solution for small enough time interval (*local existence*) is usually proved by a fixed point method. By making use of the dispersion properties of the linear operator it is shown that this equation defines a contraction in a suitable Banach space of functions for small enough time t. The local regularizing properties of the Schrödinger operator can be used to lower the regularity assumptions on the data. Existence for all time (*global existence*) holds in the case where the local solution can be continued in the large in time by means of *a priori* estimates for the norms of the solution in the corresponding spaces, resulting from conservation laws and Sobolev-type inequalities. One can then ask about *global regularity* : does the global solution preserve the regularity properties of the initial data ?

The basic estimates that are usually available are conservation of energy, decay estimates of free solutions as time increases to infinity and Strichartz-type inequalities which show that in addition to decay of the solution a small gain of special regularity may occur for positive time. Local smoothing effects have also been important in studying the nonlinear Schrödinger equation for a large class of nonlinearities including (first order) derivatives ([**22**] [**23**]). They also lead to analyticity properties in both space and time for regular initial data rapidly decaying at infinity (e.g. [**8**] [**9**]). Deep results from harmonic and Fourier analysis and sometimes the sharpening of these results have played an important role in the recent developments in studying these equations. The methods however are quite different in spirit from the ones in elliptic theory. In studying the Schrödinger map and wave map equations, gauge theoretic geometric methods need also to be incorporated to the above.

1. The Schrödinger Map problem

Schrödinger maps have been less studied than their wave map counterparts. We are interested in the study of well-posedness of the Cauchy problem with data in the Sobolev spaces $H^s(\mathbb{R}^n)$. Schrödinger maps have a natural scale invariance whence $||\cdot||_{\dot{H}^{n/2}}$ is the *critical* or *scale invariant* norm. One expects well posedness for the Cauchy problem in the subcritical case; i.e. when the data is in the Sobolev space H^s with $s > n/2$.

In their ferromagnetic formulation for maps into the sphere $\mathbb{S}^2$, smooth solutions were found by C. Sulem, J.P. Sulem and C. Bardos [**44**]. They established the local existence and uniqueness of (S) for initial data in $H^s(\mathbb{R}^n)$, $s > n/2+1$, $n \geq 2$ by the difference method. More recently, Ding and Wang [**13**][**14**] show existence and uniqueness of local smooth solutions of Schrödinger flows from $\mathbb{R}^n$ or a compact Riemannian manifold into a complete Kähler manifold. Thus obtaining local well-posedness of the Heisenberg spin system with data in H^s, $s > n/2+1$.

Chang, Shatah and Uhlenbeck were able to handle the one dimensional case for arbitrary surfaces using a generalized Hasimoto transformation [**7**] [**17**]. They also obtained the radially symmetric case when $n = 2$ in H^1, the energy norm, and assuming small energy. The symmetry assumptions allow them to gauge away the derivative nonlinearities leaving a polynomial or power nonlinearity.

Y. Ishimori [**19**] introduced a generalization of the Heisenberg model and of the Landau-Lifschitz equation. Ishimori system is intrinsically related to (1). Work in this direction has been pursued by Kenig, Ponce and Vega among others (c.f. [**24**] and references therein)

1.1. Formulation of the problem and results.

Consider then the Schrödinger map equation

$$\frac{\partial s}{\partial t} = \pm i \sum_j \nabla_j \frac{\partial s}{\partial x_j} \tag{1}$$

By relying on the work by Uhlenbeck [**50**] our first result establishes the the existence of a 'good gauge'(c.f [**33**]). Roughly speaking, a gauge transformation is a very special change in coordinates twisting the map and the connection while preserving the equation. Thus we next change gauge in this equation from the coordinate frame of the stereographic projection to a normalized frame, and rotate the frame to put the pull-back covariant derivative ∇_j as near to the 'flat' derivative $\frac{\partial}{\partial x_j}$ as possible. Schematically:

$$\nabla_j \to \mathcal{D}_j = (1+|s|^2)e^{i\psi} \circ \nabla_j \circ (1+|s|^2)e^{-i\psi} = \frac{\partial}{\partial x_j} + i a_j$$

where a_j is *minimized* by ψ, the Hodge gauge. The gauge change can also be chosen so that $div\ a = 0$ (c.f. [**33**]).

Since we will lose track of the map s during this process, we will need a set of consistency equations, which would be needed to recover the map s. So in addition to the Schrödinger map equation we have two sets of consistency conditions:

$$\nabla_j \frac{\partial s}{\partial x_k} = \nabla_k \frac{\partial s}{\partial x_j} \qquad j = 0, 1, \dots, n \quad k = 1, \dots, n \tag{2}$$

This follows from the fact that the Levi-Civita connection on $\mathbb{S}^2$ has no torsion.

$$[\nabla_j, \nabla_k] = -4i\ Im(\overline{b_j} b_k) \qquad j = 0, 1, \dots, n \quad k = 1, \dots, n \tag{3}$$

where

$$b_j = \frac{\partial s}{\partial x_j} / (1+|s|^2).$$

This follows from the fact that the curvature of $\{\nabla_j\}$ is the pull-back of constant curvature on $\mathbb{S}^2$ by the map s. The appearance of $(1+|s|^2)$ is due to stereographic projection. Observe that the consistency conditions (3) and (4) include the $t = x^0$ direction. Schematically now we have :

$$s \to b \to u \qquad\qquad \frac{\partial s}{\partial x_j} \to u_j = (1+|s|^2)^{-1} e^{i\psi} \frac{\partial s}{\partial x_j}$$

As a consequence of above procedure one obtains a *derived subset* of the complete (overdetermined) system of equations (1)-(3) which forms a well posed NLS flow and is satisfied by $\{u_j\}$ for $j = 1, \dots n$

$$\frac{\partial u_j}{\partial t} = i\Delta u_j \ - \ 2\sum_k a_k \frac{\partial u_j}{\partial x_k} \ - \ \left(\sum_k a_k^2\right) u_j + 2\ Im(\overline{u_j} u_k) u_j - i a_0 u_j \tag{4}$$

where

$$
\begin{aligned}
a_k &= \sum_{l=1}^{n} \frac{\partial \kappa_{lk}}{\partial x_l}; \qquad\qquad d\ \kappa = 0;\\
\Delta\kappa_{kj} &= -4\ Im(u_k\overline{u_j}) \qquad j=0,1,\ldots,n;\ k=1,\ldots,n\\
\Delta a_0 &= -4\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}Re(u_k\overline{u_j}) - \frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)^2 u_j\overline{u_j}\right]
\end{aligned}
$$

We call this subset of equations the *modified Schrödinger map* (MSM) system.

Remarks.

1. The point of the gauge change is to *globally* change the form of the nonlinearity, which is determined by geometric considerations. As a consequence, one obtains a derivative nonlinearity which is real, smoother and has structure (i.e. *null form*).
2. The estimates we obtain are in a coordinate (gauge) system which is dependent on the map but independent of any coordinate choice.
3. It is not possible to go back *directly* from solutions of the MSM system to Schrödinger maps. One must follow instead a more circuitous route assuming the local wellposedness for data in H^k, k large for the Schrödinger map directly and proceed via an approximation argument. As consequence one can thus obtain from our Main Theorem below the local well posedness for the Schrödinger map for data in $H^{1+\epsilon}$.

According to our reductions (4) and the last remark we are then left to consider the system of coupled nonlinear Schrödinger equations in R^{2+1}

$$
(5)\begin{cases}
\partial_t u_1 &= i\Delta u_1 + 2\left(\dfrac{\partial\beta}{\partial x_1}\dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial\beta}{\partial x_2}\dfrac{\partial u_1}{\partial x_1}\right) - i\alpha u_1 - i|\nabla\beta|^2 u_1 \pm Im(u_2\overline{u_1})u_2,\\
\partial_t u_2 &= i\Delta u_2 + 2\left(\dfrac{\partial\beta}{\partial x_1}\dfrac{\partial u_2}{\partial x_2} - \dfrac{\partial\beta}{\partial x_2}\dfrac{\partial u_2}{\partial x_1}\right) - i\alpha u_2 - |\nabla\beta|^2 u_2 \pm Im(u_1\overline{u_2})u_1\\
u_1(x,0) &= u_0^1(x)\\
u_2(x,0) &= u_0^2(x).
\end{cases}
$$

where

$$
\begin{aligned}
\Delta\beta &= \pm 2 Im(u_1\overline{u_2}),\\
\Delta\alpha &= \pm\sum_{k,j=1}^{2} 2\left(\partial_{x_k}\partial_{x_j}Re(u_k\overline{u_j}) - \partial_{x_k}^2|u_j|^2\right).
\end{aligned}
$$

There has been much interest and ground-breaking work within the last decade in the study of nonlinear Schrödinger equations with general derivative nonlinearities. There is an extensive literature in the topic which we don't attempt to survey. We refer instead the reader, to the work of Kenig, Ponce and Vega [**23**] [**25**] [**22**]; Bourgain [**3**] [**4**] [**5**]; H. Chihara [**8**] [**9**], Hayashi and Ozawa [**18**]; and all references therein, for an understanding of the results and various techniques up to this point in studying the low regularity Cauchy problem in the Sobolev spaces for this type of general nonlinear Schrödinger equations. An excellent summary of some of the historical development of the subject can be found in [**23**] pages 489-491.

Remarks.

1. By *scaling and dimensional analysis*, it is easy to see that if $u(x,t)$ solves the initial value problem (5), then $u_\gamma(x,t)=\gamma u(\gamma x,\gamma^2 t)$ solves the same with data $\gamma u_0(\gamma x)$. Hence the system (5) is *critical* in $L^2(\mathbb{R}^2)$ (i.e. the critical index is $s_c=0$).
2. The terms $i\alpha u_k$ and $\pm i Im(u_k\overline{u_j})u_k$ have approximately the structure of a *cubic nonlinearity* $|u|^2u$. These have been extensively studied and estimates have been obtained in various Lebesgue and Besov spaces. In the one dimensional case ($n=1$) there are also estimates in the $X_{s,b}$ spaces [**24**]. The methods in [**11**] and [**45**] allow us to estimate in the Schrödinger $X_{s,b}$ spaces in higher dimensions as well. We will refer to these terms as F_{cubic}.
3. The term $|\nabla\beta|^2 u_j$ is essentially quintic in u. However it scales and behaves like a cubic nonlinearity. General quintic nonlinearities in two dimensions are in general handled in spaces with at least *half* derivative on the data and have critical index $s_c=1/2$ [**6**]. Nonetheless our quintic term is very special and as we shall see later, two of the entries in the five-linear forms come with a "missing derivatives'. Thus it actually has $s_c=0$. This allows us to use Sobolev embedding in conjunction with our new embedding theorem for $X_{s,b}$ spaces to get the estimates with minimal smoothness assumptions. We refer to these as $F_{quintic}$. More specifically,
$$F_{quintic}(u_1,u_2,u_3,u_4,u_5)=\nabla\Delta^{-1}(u_1\overline{u_2})\nabla\Delta^{-1}(u_3\overline{u_4})u_5.$$
4. For the first term, more refined analysis is needed, since it involves derivatives of the solution. We need to control this expression with virtually no regularity present. If for a moment one were to think of this term as $\nabla^{-1}u\dot{\nabla}v$, one might naively be tempted to hope this somehow behaves like $u\dot{v}$. But of course, this is not the case. When high frequencies v_{high} (made worse by ∇) interact with low frequencies u_{low} (made worse by ∇^{-1}) this term can be almost as bad as a derivative nonlinearity. On the other hand, the first term possesses some extra structure. Indeed to successfully control this term one needs to exploit its 'null form' structure, as we will explain in a moment. Observe that the "null form" nonlinearity is also (anti) trilinear. We call it $F_{null}(u_1,u_2,u_3)$, where
$$F_{null}(u_1,u_2,u_3)=\frac{\partial\beta}{\partial x_1}\frac{\partial u_3}{\partial x_2}-\frac{\partial\beta}{\partial x_2}\frac{\partial u_3}{\partial x_1}.$$

Our main theorem asserts that the system (5) is locally well-posed:

MAIN THEOREM. *For every $\varepsilon>0$ and data $u_0\in H^{100\varepsilon}$, there exists $T=T(\|u_0\|_{H^{100\varepsilon}})$, such that the system (5) has a unique solution u satisfying*
$$u\in C([0,T],H^{100\varepsilon})\bigcap X_{100\varepsilon,1/2+\varepsilon}.$$
Moreover there exists constant C_ε, independent of u_0 such that
$$\|u\|_{X_{100\varepsilon,1/2+\varepsilon}}\leq C_\varepsilon\|u_0\|_{H^{100\varepsilon}}$$
Finally, the map $u_0\ni H^{100\varepsilon}\to u\ni C([0,T],H^{100\varepsilon})\bigcap X_{100\varepsilon,1/2+\varepsilon}$ is Lipschitz.

We will not distinguish between u_1 and u_2 as they come in our formulae, as we will only use their functional analytic properties. Occasionally, we will be refering to the vector $u=(u_1,u_2)$ and the data $u_0=(u_0^1,u_0^2)$.

By the Duhamel's principle one obtains the following equivalent integral formulation for the system

$$u(x,t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-\tau)\Delta}F_u(\tau,\cdot)d\tau, \tag{6}$$

where F is the nonlinearity consisting of four terms in (5). Introduce the (Schrödinger version) of the *global* Bourgain spaces $X_{s,b}$ [**4**], [**5**] as the set of all functions u with

$$\int |\hat{u}(\xi,\tau)|^2 < \tau - |\xi|^2 >^{2b} < \xi >^{2s} \, d\xi d\tau < \infty,$$

where $< \xi > := (1+|\xi|^2)^{1/2}$ and $< \tau - |\xi|^2 > := (1+|\tau-|\xi|^2|^2)^{1/2}$. We also introduce the space $X^-_{s,b}$ as

$$X^-_{s,b} := \left\{ u : \int |\hat{u}(\xi,\tau)|^2 < \tau + |\xi|^2 >^{2b} < \xi >^{2s} \, d\xi d\tau < \infty \right\}.$$

Note that the dual space to $X_{s,b}$ is $X^-_{-s,-b}$. Sometimes we will refer to s as the amount of *elliptic smoothness* in $X_{s,b}$ and to b as the *Schrödinger smoothness* for the corresponding space. We will also need a *local* version of the $X_{s,b}$ spaces, since our solutions are local in nature. Define

$$\|u\|_{X_{s,b}([0,T]\times\mathbb{R}^2)} = \inf\left\{\|U\|_{X_{s,b}} : U|_{[0,T]\times\mathbb{R}^2)} = u\right\}.$$

It is well known (see for example [**40**]) that the Schrödinger semigroup $e^{it\Delta}$ has certain smoothing effect on the Schrödinger derivatives. More precisely, let ψ be a smooth characteristic function of the interval $(-1,1)$ and $1-\varepsilon > b > 1/2$ for some positive $\varepsilon > 0$. Then

$$\|\psi(\delta^{-1}t)\int_0^t e^{i(t-\tau)\Delta}F(\tau,\cdot)d\tau\|_{X_{s,b}} \leq C_\varepsilon \delta^\varepsilon \|F\|_{X_{s,b-1+\varepsilon}} \tag{7}$$

The following estimate for the growth of the free solution $e^{it\Delta}u_0$ in the Bourgain spaces is also well-known (see [**40**], Lemma 2.3.1). For $s \geq 0$ and $b > 1/2$, $0 < \delta_0 \leq 1$, we have

$$\|\psi(t/\delta_0)e^{it\Delta}u_0\|_{X_{s,b}} \lesssim \delta_0^{(1-2b)/2}\|u_0\|_{H^s} \tag{8}$$

The approach for solving (5) is by the method of Picard iterations. Therefore, to prove short-time existence (and uniqueness in certain class), one needs to show that the map

$$\Phi(v) = \psi(t/\delta_0)e^{it\Delta}u_0 + \psi(\delta^{-1}t)\int_0^t e^{i(t-\tau)\Delta}F_v(\tau,\cdot)d\tau,$$

is a contraction on a ball in a suitable Banach space. We choose $X_{100\varepsilon,1/2+\varepsilon} \times X_{100\varepsilon,1/2+\varepsilon}$, if our data $u_0 \in H^{100\varepsilon} \equiv X_{100\varepsilon,0}$. To that effect it will suffice to prove

$$\|F_{cubic}(u_1,u_2,u_3)\|_{X_{100\varepsilon,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{100\varepsilon,1/2+\varepsilon}}\|u_2\|_{X_{100\varepsilon,1/2+\varepsilon}}\|u_3\|_{X_{100\varepsilon,1/2+\varepsilon}} \tag{9}$$

$$\|F_{null}(u_1,u_2,u_3)\|_{X_{100\varepsilon,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{100\varepsilon,1/2+\varepsilon}}\|u_2\|_{X_{100\varepsilon,1/2+\varepsilon}}\|u_3\|_{X_{100\varepsilon,1/2+\varepsilon}} \tag{10}$$

and

$$\|F_{quintic}(u_1,u_2,u_3,u_4,u_5)\|_{X_{100\varepsilon,1/2+\varepsilon}} \lesssim \prod_{j=1}^{5} \|u_j\|_{X_{100\varepsilon,1/2+\varepsilon}} \tag{11}$$

Then (7) (8) (9) (10) and (11) imply that

$$\Phi : B_{R_0}(X_{100\varepsilon,1/2+\varepsilon}) \times B_{R_0}(X_{100\varepsilon,1/2+\varepsilon}) \to B_{R_0}(X_{100\varepsilon,1/2+\varepsilon}) \times B_{R_0}(X_{100\varepsilon,1/2+\varepsilon})$$

is a contraction mapping for suitably chosen R_0 and δ. Note in particular we have estimates on the differences. Moreover we will show that once we have a local existence result for data $u_0 \in H^{100\varepsilon}$ with lifespan for the solution $T = T(\|u_0\|_{H^{100\varepsilon}})$, then we have local existence for data $u_0 \in H^k$ with a lifespan for the solution $T_k = T(\|u_0\|_{H^{100\varepsilon}}, k)$. In [**9**] H. Chihara showed that for general $F(u, \partial u)$ that behave like cubic near $(0,0)$ one has local existence for data in $H^m(\mathbb{R}^2)$ provided $m \geq 5$. In a similar vein, we prove,

Theorem (Regularity estimates).
For a given data $u_0 \in H^k$, the system (5) has an unique solution u defined at least for time $T = T(\|u_0\|_{H^{100\varepsilon}}, k)$ and there exists a constant $C_{\varepsilon,k}$, so that

$$\|u\|_{X_{k,1/2+\varepsilon}} \leq C_{\varepsilon,k}\|u_0\|_{H^k}.$$

1.2 Some Generalities on Multilinear Forms.

We follow [**45**] to introduce somewhat more general framework for the multilinear forms that we have to deal with.

For an integer k and $Z = \mathbb{R}^{2+1}$ (or any abelian group) define the *hyperplane*:

$$\Gamma_k(Z) = \{(\xi_1, \dots, \xi_k) \in Z^k : \xi_1 + \ldots + \xi_k = 0\}.$$

A $[k, Z]$ multiplier is a function $m : \Gamma_k(Z) \to \mathbb{C}$, so that there exists a constant C, such that the inequality

$$\Big|\int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_j(\xi_j)\Big| \leq C \prod_{j=1}^{k} \|f_j\|_{L^2(Z)},$$

holds for all $f_1, \dots f_k \in L^2(Z)$. The best constant C with the above property is naturally called a multiplier norm for m and is usually denoted $\|m\|_{[k,Z]}$. We will also need the notation $\Gamma_k(Z, \xi_j = \eta_j) := \Gamma_k(Z) \cap \{\xi_j = \eta_j\}$ for a fixed η_j.

To get a bit of the flavor of the notation let's state two useful results (cf. [**45**]). First, by the Cauchy-Schwarz inequality, if m is a $[k, Z]$ multiplier, then

$$\|m\|_{[k,Z]} \leq \sup_{\eta_j \in Z} \left(\int_{\Gamma_k(Z,\xi_j=\eta_j)} |m(\xi)|^2 \right)^{1/2}.$$

Next, in the cases $k = 3, 4$ and for any subsets A, B, C of Z, we have the following useful estimates

$$\|\chi_A(\xi_1)\chi_B(\xi_2)\|_{[3,Z]} \leq |\{\xi_1 \in A : \xi - \xi_1 \in B\}|^{\frac{1}{2}}$$
$$\|\chi_A(\xi_1)\chi_B(\xi_2)\chi_C(\xi_3)\|_{[4,Z]} \leq |\{(\xi_1, \xi_2) \in A \times B : \zeta \in C\}|^{\frac{1}{2}}$$

where $\zeta = \xi - \xi_1 - \xi_2$ for some $\xi \in Z$.

Two other important technical tools that are crucial for us are a form of the *Schur's test* and the *box localization method* ([**45**]). We state the latter when $m(\xi_1, \dots, \xi_s) = \prod_{j=1}^{s} m_j(\xi_j)$.

LEMMA (Tao, [**45**]). *Let R be a rectangular box in Z. Suppose also that $supp_j(m)$ is contained in a $R + \eta_j$ for some $\eta_j \in Z$ and $1 \leq j \leq k-2$. Then*

$$\|m\|_{[k,Z]} \sim \sup_{\eta_{k-1}, \eta_k \in Z} \|m \chi_{R+\eta_{k-1}}(\xi_{k-1}) \chi_{R+\eta_k}(\xi_k)\|_{[k,Z]}.$$

In particular, the *box localization principle* states, that if we have a multiplier in which all but two of the variables are restricted to sets of certain diameter, we can restrict (at the expense of a bigger constant) the remaining two variables to sets with the *same diameter.*

Next, consider a fixed smooth cutoff ψ supported around 1. Introduce also ψ_0 supported around 0 with

$$\psi_0(\cdot) + \sum_{R \geq 1} \psi(R^{-1}(\cdot)) \equiv 1, \qquad \text{on} \qquad \mathbb{R}.$$

Let $\widehat{u_R} = \widehat{S_R u} = \psi(R^{-1}|\cdot|)\widehat{f}(\cdot)$ be the standard Littlewood-Paley operator in space at frequency R applied to the function $u(\cdot, t)$. We will denote by f_R the restriction of the function $f(\xi)$ to the annulus $\{\xi : |\xi| \sim R\}$.

In order to prove our results we also need :

1. Sharp estimates of Tao for the $[3, Z]$ and $[4, Z]$ norms of multipliers like those in Corollary above for sets A, B or C that typically look like $\tau_i - |\xi_i|^2 \sim L_i$, $|\xi_i| \sim R_i$ or $|\xi_i - \zeta_0| \sim r$ and $i = 1, 2$ or 3.
2. The following adaptation of a result due to Colliander-Delrot-Kenig and Staffilani [**11**].

LEMMA 1. *The trilinear form $C(f, g, h)$ defined by*

$$\int_{\Gamma_3(R^{2+1})} \frac{f_R(\xi_1, \tau_1) g_M(\xi_2, \tau_2)}{< \tau_2 - |\xi_2|^2 >^{1/2+\varepsilon}} \frac{h_N(\xi_3, \tau_3)}{< \tau_3 + |\xi_3|^2 >^{1/2+\varepsilon}}$$

is bounded -up to a constant- by

$$\left(\frac{\min(M, N))}{\max(M, N)}\right)^{1/2} \|f_R\|_{L^2(\mathbb{R}^3)} \|g_M\|_{L^2(\mathbb{R}^3)} \|h_N\|_{L^2(\mathbb{R}^3)} \tag{12}$$

As a corollary, one has that $\|S_R((u_1)_M(\overline{u_2})_N)\|_{L^2(\mathbb{R}^3)}$ is bounded -up to a constant- by

$$\left(\frac{\min(M, N))}{\max(M, N)}\right)^{1/2} \max(M, N)^{-s} \Pi_{i=1}^2 \|u_i\|_{X_{s,1/2+\varepsilon}}$$
$$\lesssim \left(\frac{\min(M, N))}{\max(M, N)}\right)^{1/2} R^{-s} \|u_1\|_{X_{s,1/2+\varepsilon}} \|u_2\|_{X_{s,1/2+\varepsilon}}$$

Note that $\max(M,N) \gtrsim R$, since otherwise

$$\operatorname{supp} \mathcal{F}((u_1)_M(\overline{u_2})_N) \subset \{\xi : |\xi| \ll R\}$$

and $S_R((u_1)_M(\overline{u_2})_N) = 0$.

1.3 Cubic and Quintic Nonlinearities.

Armed with these we can now prove the following result. This will also play a role in the *off diagonal* considerations for the “null form” nonlinearity.

PROPOSITION 2. *Let m be a bounded function. Then for $1/2 > \kappa > 5\varepsilon$, the quatrilinear form $H(f,g,h,w)$ defined by*

$$\int_{\substack{\xi_1+\xi_2+\xi_3+\xi_4=0,\\ \tau_1+\tau_2+\tau_3+\tau_4=0}} m(\xi_i) \; \frac{f(\xi_1,\tau_1) <\xi_1>^{-\kappa}}{<\tau_1 - |\xi_1|^2>^{1/2+\varepsilon}} \; \frac{g(\xi_2,\tau_2) <\xi_2>^{-\kappa}}{<\tau_2 + |\xi_2|^2>^{1/2+\varepsilon}} \times$$

$$\times \frac{h(\xi_3,\tau_3) <\xi_3>^{\kappa}}{<\tau_3 - |\xi_3|>^{1/2-2\varepsilon}} \; \frac{w(\xi_4,\tau_4) <\xi_4>^{-\kappa}}{<\tau_4 \pm |\xi_4|>^{1/2+\varepsilon}} d\xi_1 \ldots d\tau_1 \ldots$$

satisfies

$$|H(f,g,h,w)| \lesssim \|f\|_2 \|g\|_2 \|h\|_2 \|w\|_2.$$

We need the nontrivial m to cover some cases where we have a *zero order* pseudo-differential operator acting on some of the entries.

By Cauchy-Schwarz the estimate for this quatrilinear form is reduced to the product of two trilinear forms. The estimates for the trilinear forms are obtained by relying on Colliander, Delrot, Kenig and Staffilani's result [**11**] and the estimates of Tao in [**45**] for the $[3,Z]$ and $[4,Z]$ multipliers. The idea is to perform a Littlewood-Paley decomposition in the elliptic and ‘Schrödinger’ directions, applying Lemma 1 in conjunction with Tao's estimates depending on the relative sizes of the ‘Schrödinger’ weights.

The cubic-like nonlinearities such as (9) now follow from the following result which in turn is a consequence of Proposition 2.

LEMMA 3. *Let $s > 5\varepsilon$ then we have*

$$\|u_1\overline{u_2}u_3\|_{X_{s,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{s,1/2+\varepsilon}} \|u_2\|_{X_{s,1/2+\varepsilon}} \|u_3\|_{X_{s,1/2+\varepsilon}}$$
$$\|u_1\overline{u_2}\overline{u_3}\|_{X_{s,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{s,1/2+\varepsilon}} \|u_2\|_{X_{s,1/2+\varepsilon}} \|u_3\|_{X_{s,1/2+\varepsilon}}$$
$$\|u_1 u_2 u_3\|_{X_{s,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{s,1/2+\varepsilon}} \|u_2\|_{X_{s,1/2+\varepsilon}} \|u_3\|_{X_{s,1/2+\varepsilon}}$$

Quintic nonlinearities are difficult to control in a space with less than a full derivative. We have however a very special form of the nonlinearity, which makes it tractable. A good model expression of what we are dealing with is

$$|\nabla|^{-1}(u_1\overline{u_2})|\nabla|^{-1}(u_3\overline{u_4})u_5,$$

where $|\nabla|^{-1}$ is a smoothing pseudodifferential operator of order -1. At the first step, we use the cubic estimates outlined above and then we exploit the “smoothing” provided by $|\nabla|^{-1}$ via Sobolev embedding. To accomplish this program, we need to pass from $X_{s,b}$ to a mixed norm Lebesgue spaces and back. It is also a question of independent interest to study the relation between the $X_{s,b}$ spaces and the more classical mixed norm spaces. Actually, the reader is probably aware that most of the current existence results are in fact proved via an appropriate contraction mapping argument in mixed Lebesgue spaces of various sorts. The mixed space-time Lebesgue spaces are defined as the set of all functions u with

$$\|u\|_{L^p_t L^q_x} = \left(\int \left(\int |u(x,t)|^q dx \right)^{p/q} dt \right)^{1/p}.$$

The next lemma supplies an embedding theorem for $X_{s,b}$ spaces into a mixed norm Lebesgue spaces. In the case $p = 2$ the lemma below is a well-known result by Bourgain.

LEMMA 4. *For $1 \leq p \leq 2$,*

$$X_{0,1/2+} \hookrightarrow L^{2p'}_t L^{2p}_x(\mathbb{R}^2 \times \mathbb{R}) \tag{13}$$

More generally, we have the bilinear estimates

$$\|uv\|_{L^{p'}_t L^p_x} \lesssim \|u\|_{X_{0,1/2+\varepsilon}} \|v\|_{X_{0,1/2+\varepsilon}} \tag{14}$$

$$\|u\overline{v}\|_{L^{p'}_t L^p_x} \lesssim \|u\|_{X_{0+,1/2+\varepsilon}} \|v\|_{X_{0+,1/2+\varepsilon}} \tag{15}$$

The embedding (13) actually holds in more generality and can be proven from the Strichartz estimates. However, the above suffices for our purposes. Besides, we actually need (14) and (15).

We now sketch the idea of the proof and estimates required for the quintic nonlinearities. To obtain the estimate for $\||\nabla\beta|^2 u_j\|_{X_{100\varepsilon,-1/2+2\varepsilon}}$ in (11) we first perform a dyadic decomposition on $\nabla\beta$ and $\nabla\overline{\beta}$ to get -say- for $j = 1$

$$|\nabla\beta|^2 u_1 = \sum_{r_1,r_2} S_{r_1}(\nabla\beta) S_{r_2}(\nabla\overline{\beta}) u_1.$$

We then split into two pieces, $\max(r_1, r_2) \leq 1$ and $\max(r_1, r_2) \geq 1$. To illustrate how estimates (14)-(15) come in let us sketch the argument in the *small frequency* case. Observe that

$$S_{r_1}(\nabla\beta) S_{r_2}(\nabla\overline{\beta}) = S_{\max(r_1,r_2)}(S_{r_1}(\nabla\beta) S_{r_2}(\nabla\overline{\beta})).$$

Therefore, by the same argument used to estimate the trilinear forms in the proof of Proposition 2, we obtain:

$$\sum_{r_1,r_2 \leq 1} \|S_{r_1}(\nabla\beta) S_{r_2}(\nabla\overline{\beta})\, u_1\|_{X_{100\varepsilon,-1/2+2\varepsilon}} \lesssim$$
$$\lesssim \|\nabla\beta\|^2_{L^4(\mathbb{R}^2\times\mathbb{R})} \|u_1\|_{X_{100\varepsilon,1/2+\varepsilon}}$$

To estimate $\|\nabla\beta\|_{L^4(\mathbb{R}^2\times\mathbb{R})}$, the Sobolev embedding with one derivative in the space variable yields

$$\|\nabla\beta\|_{L^4_x(\mathbb{R}^2)}\| \lesssim \|\nabla^2\beta\|_{L^{4/3}_x(\mathbb{R}^2)}$$

From the definition of β, the boundedness of the Riesz transforms and the bilinear estimate (15), we have

$$\|\nabla^2\beta\|_{L_t^4 L_x^{4/3}} \lesssim \|\nabla^2\Delta(u_1\overline{u_2})\|_{L_t^4 L_x^{4/3}} \lesssim$$
$$\lesssim \|u_1\overline{u_2}\|_{L_t^4 L_x^{4/3}} \lesssim \Pi_{i=1}^2 \|u_i\|_{X_{\varepsilon,1/2+\varepsilon}}.$$

Thus

$$\|\nabla\beta\|_{L^4(\mathbb{R}^2\times\mathbb{R})} \lesssim \|u_1\|_{X_{\varepsilon,1/2+\varepsilon}} \|u_2\|_{X_{\varepsilon,1/2+\varepsilon}}$$

1.4 Derivative Nonlinearities with Structure .

This section is the heart of the matter. First, note that in the "null form" nonlinearity F_{null} the components look identical except for the dependence upon u_1 and u_2, which is irrelevant in our argument. Thus, it suffices to consider only one component. We test the "null form" against a function $W \in X^-_{-100\varepsilon,1/2-2\varepsilon}$ to get the (anti) multilinear form

$$M(u_3,u_1,u_2,W) = \int \left(\frac{\partial\beta}{\partial x_1}\frac{\partial u_3}{\partial x_2} - \frac{\partial\beta}{\partial x_2}\frac{\partial u_3}{\partial x_1}\right)\overline{W}dxdt.$$

where $\Delta\beta = \operatorname{Im}(u_1\overline{u_2})$. Thus our goal now is to show the following bound

THEOREM 5.

$$|M(u_1,u_1,u_3,W)| \lesssim \Pi_{i=1}^3 \|u_i\|_{X_{100\varepsilon,1/2+\varepsilon}} \|W\|_{X_{-100\varepsilon,1/2-2\varepsilon}}. \tag{16}$$

Integration by parts in the definition of M allow us to take the derivatives off β. The special cancelation properties of the expression and the lack of boundary terms allow us to rewrite M as

$$M(u_3,u_1,u_2,W) = \int \beta\left(\frac{\partial\overline{W}}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial\overline{W}}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right).$$

From $\Delta\beta = u_1\overline{u_2}$, we further have

$$M(u_3,u_1,u_2,W) = \int u_1\overline{u_2}G, \tag{17}$$

where

$$G = \Delta^{-1}\left(\frac{\partial\overline{W}}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial\overline{W}}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right).$$

Parseval's identity, together with

$$\widehat{\overline{u_2}}(\xi,\tau) = \widehat{\overline{u_2}}(-\xi,-\tau)$$

imply that

$$M(u_3,u_1,u_2,W) = \int \widehat{u_1 G}(\xi,\tau)\widehat{\overline{u_2}}(-\xi,-\tau)d\xi d\tau$$

Since the complex conjugation is an isometry of $X_{s,b}$ onto $X^-_{s,b}$, we can write

$$\widehat{\overline{u_2}}(-\xi,-\tau) = \frac{h_2(-\xi,-\tau)}{<\tau-|\xi|^2>^{1/2+\varepsilon}} <\xi>^{-100\varepsilon},$$

for some $h_2 \in L^2$. Similarly by using the properties of $X_{s,b}$ spaces, we express u_1 and u_2 via its L^2 representative h_1 and h_2 (respectively) and some weights dictated by the particular space. All in all we get that $M(u_3, u_1, u_2, W)$ equals

$$\int \frac{h_1(\xi-\eta, \tau-\mu) <\xi-\eta>^{-100\varepsilon}}{<\tau-\mu-|\xi-\eta|^2>^{1/2+\varepsilon}} \widehat{G}(\eta,\mu) \ \frac{h_2(-\xi,-\tau)<\xi>^{-100\varepsilon}}{<\tau-|\xi|^2>^{1/2+\varepsilon}} d\xi d\eta d\tau d\mu,$$

where

$$\begin{aligned}\widehat{G}(\eta,\mu) &= \frac{1}{|\eta|^2}\int (\eta_1 z_2 - \eta_2 z_1)\,\widehat{\overline{W}}(\eta - z, \mu - s)\,\widehat{u_3}(z,s) dz ds\\ &= \frac{1}{|\eta|^2}\int \langle \eta\,,\, z^{\perp}\rangle\,\widehat{\overline{W}}(\eta - z, \mu - s)\,\widehat{u_3}(z,s) dz ds\end{aligned}$$

We need the representation of M as a quatrilinear form applied to four $L^2(\mathbb{R}^{2+1})$ functions. Hence by a change variables we obtain the *quatrilinear form*

$$\begin{aligned}\Lambda(h_1,h_2,f,g) \ = &\\ &\int \frac{h_1(\xi-\eta+z,\tau-\mu)<\xi-\eta+z>^{-100\varepsilon}}{<\tau-\mu-|\xi-\eta+z|^2>^{1/2+\varepsilon}}\ \frac{h_2(-\xi,-\tau)<\xi>^{-100\varepsilon}}{<\tau-|\xi|^2>^{1/2+\varepsilon}}\times\\ &\times\frac{\langle \eta\,,\, z^{\perp}\rangle}{|\eta-z|^2}\frac{f(\eta,\mu-s)<\eta>^{100\varepsilon}}{<\mu-s-|\eta|^2>^{1/2-2\varepsilon}}\ \frac{g(-z,s)<z>^{-100\varepsilon}}{<s-|z|^2>^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds\end{aligned}$$

We break up the η and z integration in the definition of Λ to obtain

$$\begin{aligned}\Lambda(h_1,h_2,f,g) &= \int_{|z|\le|\eta|/2 \quad \& \quad |\eta|\le|z|/2} ... + \int_{|z|\sim|\eta|} ...\\ &= \Lambda_{off\,diag} + \Lambda_{diag}\end{aligned}$$

and estimate each term separately.

Estimates away from the diagonal.

This is the case when the "null" form is under control in the L^∞ norm. By symmetry, it suffices to consider the case $|z|\le|\eta|/2$. Thus, we have the estimate

$$\frac{\langle \eta\,,\, z^{\perp}\rangle}{|\eta|^2} \lesssim \frac{|z|}{|\eta|}$$

On the other hand

$$\Lambda_{offdiag} = \sum_R \int S_R(u_1\overline{u_2})S_R(\tilde{G}) = \sum_{M,N,R}\int S_R((u_1)_M(\overline{u_2})_N)\tilde{G}_R d\eta d\mu, \tag{18}$$

where $\widehat{\tilde{G}}(\eta,\mu)$ equals

$$\int_{|z|\le|\eta|/2} \frac{\langle \eta\,,\, z^{\perp}\rangle}{|\eta|^2}\frac{f(\eta-z,\mu-s)<\eta-z>^{100\varepsilon}}{<\mu-s-|\eta-z|^2>^{1/2-2\varepsilon}}\ \frac{g(z,s)}{<z>^{100\varepsilon}<s-|z|^2>^{1/2+\varepsilon}} dz ds.$$

Every term in the dyadic formula (18) can be estimated by corresponding term for the form H from Propostition 2 times r/R with $\kappa = 100\varepsilon$, which makes the double summation in r, R easier. We get

$$|\Lambda_{offdiag}| \lesssim \|h_1\|_2\|h_2\|_2\|f\|_2\|g\|_2.$$

Diagonal Estimates: Null forms.

First we represent the integration region over η and z as a union of dyadic annuli of the form $|\eta| \sim |z| \sim R$. Denote $\theta = \angle(\eta, z)$. If $|\theta| > 1/100$, one can control the L^∞ norm of the "null" form in a similar manner as before. So by symmetry, we assume that $0 < \theta < 1/100$. Then we decompose in the angular variable θ in a dyadic manner as $\theta \to 0$.

Observe that

$$\begin{aligned}|\eta - z|^2 &= (|\eta| - |z|)^2 + 2|\eta||z|(1-\cos\theta) \sim \\ &\sim (|\eta| - |z|)^2 + R^2\theta^2, \\ \langle \eta, z^\perp\rangle &= |\eta||z| \sin\theta \ \sim\ R^2\theta.\end{aligned}$$

Obviously the size of $|\eta| - |z|$ is important at this stage, so we partitioned further the area of integration to take this into account as follows. For every $l \geq 1$,

$$\begin{aligned}\mathcal{A}_l &= \left\{(\eta, z) : ||\eta| - |z|| \sim 2^l R\theta\right\} \\ \mathcal{A}_0 &= \{(\eta, z) : ||\eta| - |z|| \leq R\theta\}.\end{aligned}$$

The set $\mathcal{A}_0$ is the "really diagonal case". When integrating on $\mathcal{A}_l$ - the "not so diagonal case"- one can obtain similar estimates with the corresponding exponential decay in l.

In the $\mathcal{A}_0$ case, we aim at controlling the expression

(19)

$$\sum_{R,\theta} \int_{|\eta|,|z|\sim R, ||\eta|-|z||\leq R\theta} \frac{\langle \eta, z^\perp\rangle}{|\eta - z|^2} \ \frac{h_1(\xi-\eta+z, \tau-\mu) < \xi-\eta+z >^{-100\varepsilon}}{< \tau-\mu-|\xi-\eta+z|^2 >^{1/2+\varepsilon}} \times$$

$$\times \frac{h_2(-\xi,-\tau) < \xi >^{-100\varepsilon}}{< \tau - |\xi|^2 >^{1/2+\varepsilon}} \frac{f(\eta, \mu-s) < \eta >^{100\varepsilon}}{< \mu - s - |\eta|^2 >^{1/2-2\varepsilon}} \frac{g(-z,s) < z >^{-100\varepsilon}}{< s - |z|^2 >^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds.$$

Note that

$$\left|\frac{\langle \eta, z^\perp\rangle}{|\eta - z|^2}\right| \sim \theta^{-1}, \qquad \text{and} \qquad < \eta >^{100\varepsilon} < z >^{-100\varepsilon} \lesssim 1$$

whenever $(\eta, z) \in \mathcal{A}_0$.

We take next a partition the annulus $|\eta| \sim R$ into a finite intersection family of cubes with sides $R\theta$

$$\{|\eta| \sim R\} = \bigcup_\nu Q_\nu(\eta_\nu, R\theta).$$

By our assumptions on η and z, it is clear that whenever $\eta \in Q_{\nu_0}$, then $z \in Q^*_{\nu_0}$, the cube with same center and four times its sidelength. By the Schur's test ([**45**], Lemma 3.11) we have

$$\|\chi_{\{(\eta,z)\in\mathcal{A}_0\}}\|_{[4,R^{2+1}]} \lesssim \|\sum_\nu \chi_{\{\eta\in Q_\nu, z\in Q^*_\nu\}}\|_{[4,R^{2+1}]} \lesssim \sup_\nu \|\chi_{\{\eta, z\in Q^*_\nu\}}\|_{[4,R^{2+1}]}$$

Thus, we may further restrict the region of integration to a given cube $Q_0(\eta_0, R\theta)$. Since we have managed to restrict two of the variables to a box of size $R\theta$, by the box localization principle we may do so for the other two variables as well. Thus, we are lead to consider multipliers of the form

$$\chi_{\{(\eta, z\in Q_0(\eta_0, R\theta), \xi\in B(\xi_0, R\theta)\}},$$

where $|\eta_0| \sim R$ and $\xi_0 \in \mathbb{R}^2$ (η_0 and ξ_0 may depend on R, θ). Denote

$$L_1 = \tau - \mu - |\xi - \eta + z|^2, \quad L_2 = \tau - |\xi|^2, \quad L_3 = \mu - s - |\eta|^2, \quad L_4 = s - |z|^2.$$

To control (19) we need to show that

$$\sum_{R,\theta} \sup_{\eta_0, \xi_0 \in \mathbb{R}^2; |\eta_0| \sim R} \frac{\max(<|\xi_0|>, <R\theta>)^{-100\varepsilon}}{\theta} \times$$

$$\times \| \frac{\chi\{\eta, z \in Q_0(\eta_0, R\theta), |\xi - \xi_0| \leq R\theta\}}{(<L_1><L_2><L_4>)^{1/2+\varepsilon} L_3^{1/2-2\varepsilon}} \| \lesssim 1, \tag{20}$$

where we have used that $<\xi - \eta + z><\xi> \geq \max(<|\xi_0|>, <R\theta>)$.

Now, if we write the equivalent quatrilinear form $\Lambda_0(h_1, h_2, f, g)$ representing the multiplier in (20)

$$\int_{z, \eta \in Q_0(\eta_0, R\theta), |\xi - \xi_0| < R\theta} \frac{h_1(\xi - \eta + z, \tau - \mu)}{<L_1>^{1/2+\varepsilon}} \frac{h_2(-\xi, -\tau)}{<L_2>^{1/2+\varepsilon}} \times$$

$$\times \frac{f(\eta, \mu - s)}{<L_3>^{1/2-2\varepsilon}} \frac{g(-z, s)}{<L_4>^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds.$$

We actually show that

$$\sup_{\eta_0, \xi_0 : |\eta_0| \sim R} |\Lambda_0(h_1, h_2, f, g)| \leq C_{R,\theta,\xi_0},$$

for suitable C_{R,θ,ξ_0}, such that

$$\sum_{R,\theta} \sup_{\xi_0} \frac{\max(<|\xi_0|>, <R\theta>)^{-100\varepsilon}}{\theta} C_{R,\theta,\xi_0} \lesssim 1.$$

as required (c.f. [**33**] for details).

2. The Wave Map problem

The *wave map* equation between two Riemannian manifolds, which is the natural Minkowskian analogue of the harmonic map equation has been extensively studied by a number of mathematicians in the last decade. In the difficult case of two dimensions, some of the early results include those of Christodoulou and Tahvildar-Zadeh [**10**] who studied the regularity of spherically symmetric wave maps assuming a convexity condition in the target manifold. Shatah and Tahvildar-Zadeh [**38**] [**39**] studied the optimal regularity of equivariant wave maps into two-dimensional rotationally symmetric and geodesically convex Riemannian manifolds. The study of the general wave map problem incorporated methods that exploited the null-form structure of the wave map system such as that of M. Grillakis [**15**] as well as the geometric structure of the equation as done by M. Struwe [**41**] [**43**]. The work of Klainerman and Machedon [**26**] [**27**] [**28**] and of Klainerman and Selberg [**30**] gives the almost optimal local well-posedness for the general Cauchy problem in any dimensions bigger or equal then two. For a more complete presentation of the developments in this subject we refer the interested reader to the book by Shatah and Struwe [**36**].

More recently, the pivotal work of Tataru [**48**], [**49**], followed by that of Tao [**46**] [**47**] have introduced new techniques which allow one to treat the Cauchy problem with *critical* data. Their methods rely on and further develop important ideas from harmonic analysis; and in Tao's case are used in conjunction with gauge theoretic geometric methods.

2.1 Formulation of the problem and results.

Consider $\mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{1+n}$ be Minkowski space with flat metric and let G be any compact Lie group or Riemannian symmetric space (e.g. $\mathbb{S}^m$).
Let ∂_α, ∂^α $\alpha = 0, 1, \ldots, n$ be the usual derivatives with respect to Minkowski metric. Then,

$$\Box := \partial_\alpha\partial^\alpha = -\partial_t^2 + \Delta$$

is the D'Alembertian operator.

A (classical) wave map is a (smooth) map $\phi : \mathbb{R}\times\mathbb{R}^n \to G$ satisfying,

$$\Box\phi = -\phi(\partial_\alpha\phi)^T\partial^\alpha\phi$$

(ϕ constant outside the finite union of light cones). Wave maps also have a natural scaling from where the *critical index* for the Cauchy problem with data in the Sobolev spaces is $s_c = \frac{n}{2}$.

When the dimension $n \geq 2$ and the Cauchy data $(\phi, \partial_t\phi) \restriction_{t=0} = (f, g) \in H^s \times H^{s-1}$ with $s > n/2$ it is known that the (W) is locally well posed and the solution can be continued -without losing regularity- as long as the H^s-norm remains bounded [**26**] [**27**] [**28**][**30**].

When instead the Cauchy data $(\phi, \partial_t\phi) \restriction_{t=0} = (f, g) \in \dot{B}^{n/2}_{2,1} \times \dot{B}^{n/2-1}_{2,1}$ is *sufficiently small* then there is global well posedness, regularity and scattering for (W). Note that $\dot{B}^{n/2}_{2,1} \hookrightarrow L^\infty$. For $n \geq 5$ the result is in [**48**] and for $n = 2, 3, \ldots$ in [**49**].

For the question of *global regularity* asking whether solutions to (W) corresponding to smooth initial data, small in $\|\cdot\|_{s_c}$ stay smooth for all time the following results in the case of $G = \mathbb{S}^m$ and $n \geq 2$ hold.

Suppose the Cauchy data $(\phi, \partial_t\phi) \restriction_{t=0} = (f, g) \in H^s \times H^{s-1}$ with $s > n/2$ and critical norm $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ *sufficiently small*,then solutions to (W) have global regularity. Furthermore, for s close to $n/2$ have global bounds

$$\|((\phi, \partial_t\phi)\|_{L^\infty_t(H^s_x\times H^{s-1}_x)} \lesssim \|(f, g)\|_{H^s_x\times H^{s-1}_x}.$$

For $n \geq 5$ this result is in [**46**] and for $n = 2, 3, \ldots$ in [**47**].

More recently, roughly at the same time and independently, Shatah and Struwe on the one hand [**36**] and Nahmod, Stefanov and Uhlenbeck on the other [**34**] refined Tao's result for higher dimensions in [**Tao1**] as follows

MAIN THEOREM. *Let $n \geq 4$, $\phi : \mathbb{R}\times\mathbb{R}^n \to G$. Suppose the Cauchy data $(\phi, \partial_t\phi) \restriction_{t=0} = (f, g)$ has sufficiently small norm in $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$. Then there exists a <u>unique</u> <u>global</u> solution to the wave map problem such*

$$\|((\phi, \partial_t\phi)\|_{L^\infty_t(\dot{H}^{n/2}_x\times \dot{H}^{n/2-1}_x)} \lesssim \|(f, g)\|_{\dot{H}^{n/2}_x\times \dot{H}^{n/2-1}_x}.$$

Moreover, there is global regularity; i.e. if in addition $(f, g) \in H^s \times H^{s-1}$ with $s > n/2$ then the solution $(\phi, \partial_t\phi)$ belongs to $H^s \times H^{s-1}$ for all time and satisfies global bounds (as in Tao's result above).

Remarks.

1. Shatah and Struwe result is more general. In their case the target is any complete Riemannian manifold with bounded curvature.
2. Nahmod, Stefanov and Uhlenbeck prove it when the target is any compact Lie group or Riemannian symmetric space.
3. Our method combines both delicate techniques from harmonic analysis with fairly standard global gauge theoretic geometric methods. Both our work and that of Shatah-Struwe use the same gauge change; the analytic approach however, is significatly different as Shatah-Struwe base their results on Lorentz spaces and we use Besov spaces. Besov spaces are contained in Lorentz spaces -for appropriate indices-. Lorentz spaces are better behaved under coordinate transformations.
4. In none of the works above is possible to obtain (strong) well posedness at the critical level for the wave map itself. In other words even though one indeed has well posedness for the *gauged map*; there are no estimates available on *differences* for the *original wave map* itself and one cannot obtain any continuous dependence of the map on the data in the coordinate setting. It thus seems reasonable to think that the notion of wellposedness is not appropriate for this type of geometric equations at the critical level. The problem stems in that *well posedness* is not a gauge invariant notion; it is not even necessarily true that uniqueness in one coordinate system implies uniqueness in another directly.

Regard the wave map equation as an equation given through covariant derivatives. These arise as follows:

$$s : \mathbb{R} \times \mathbb{R}^n \to M$$

where M is an arbitrary Riemannian manifold and

$$ds : \mathcal{T}(\mathbb{R} \times \mathbb{R}^n) \to \mathcal{T}M$$

where $\mathcal{T}(\mathbb{R} \times \mathbb{R}^n) = (\mathbb{R} \times \mathbb{R}^n) \times (\mathbb{R} \oplus \mathbb{R}^n)$.
Let $s^*\nabla$ be the pullback of the Levi-Civita connection on M to $s^*\mathcal{T}M$ via the map s. Then, in coordinate free notation, the wave map equation is

$$s^*\nabla_0 \frac{\partial s}{\partial t} - \sum_{j=1}^{n} s^*\nabla_j \frac{\partial s}{\partial x^j} = 0 \tag{1}$$

In studying (1) we need to consider two extra sets of (consistency) equations. Once comes from the fact that the Levi-Civita connection on M is torsion free hence,

$$s^*\nabla_j \frac{\partial s}{\partial x^k} = s^*\nabla_k \frac{\partial s}{\partial x^j}\,, \tag{2}$$

for $j = 0, 1, \ldots, n\,,\ k = 1, \ldots, n$ where we have set $t = x^0$.

The other one controls the curvature of $s^*\nabla$ via the equation

$$[s^*\nabla_j, s^*\nabla_k] = R(s)(\frac{\partial s}{\partial x^j}, \frac{\partial s}{\partial x^k}) \tag{3}$$

The *wave map system* (1)-(3) is overdetermined.

Remark. We assume the map s is topologically trivial (the wave map fixes spatial infinity so topologically, $s : \mathbb{R}^n \cup \infty \to M$). Hence, $s^*\mathcal{T}M$ is the trivial bundle $(\mathbb{R}^1 \times \mathbb{R}^n) \times \mathbb{R}^m$.

Our first result asserts that under smallness assumptions on $s \in L^\infty_t \dot{W}^{1,n/2}_x$, there is a unique choice of coordinates for $s^*\mathcal{T}M$. This follows from the existence and uniqueness of a (Coulomb) *gauge change* g under suitable hypothesis on the space-time curvature $F_A = dA + [A, A]$ and connection A. More precisely we have

THEOREM (Existence of a good gauge). *Let $d + A$ be a smooth connection with compact structure groups G over $\mathbb{R}\times\mathbb{R}^n$ or $I\times\mathbb{R}^n$. Assume $A \sim 0$ at spatial infinity and let $F_A = dA + [A, A]$ be the space-time curvature. Then there exists a positive constant $\epsilon = \epsilon(n, G)$ such that if the mixed space-time Lebesgue norm*

$$\|F_A\|_{L^\infty_t L^{n/2}_x} < \epsilon\,,$$

then, there exits a unique smooth gauge change g, $g \sim I$ at spatial infinity, such that if $\tilde{A} = gAg^{-1} - dg\,g^{-1}$ we have,

1. $\|\tilde{A}\|_{L^\infty_t \dot{W}^{1,n/2}_x} \leq c(n, G)\|F_A\|_{L^\infty_t L_x{}^{n/2}}$
2. $\sum_{j=1}^n \frac{\partial}{\partial x^j}\tilde{A}_j = 0$

The method of proof follows the method used by K. Uhlenbeck in [**50**]. More to the point in our case is the following result which is an immediate consequence of the above.

COROLLARY. *The Theorem above remains true if $A \in L^\infty_t \dot{W}^{1,n/2}_x$ and $F_A \in L^\infty_t L^{n/2}_x$*

We now apply this gauge change in the wave map system (1)-(3). If we denote by $b = ds$ schematically we get :

$$b_j \to g b_j g^{-1} := \tilde{b}_j$$

$$g s^* \nabla_j g^{-1} \to \frac{\partial}{\partial x_j} + \frac{1}{2} g b_j g^{-1} - dg g^{-1} := \frac{\partial}{\partial x_j} + a_j := D_j$$

In the new gauge, with $\operatorname{div} a = 0$, we have the *same* equations (1)-(3) albeit with D_j replacing $s^*\nabla_j$ and $\tilde{b}_j, b_j$. More precisely, let $b = ds$. Then the equations themselves are written

$$D_0 \tilde{b}_0 - \sum_{j=1}^n D_j \tilde{b}_j = 0.$$

$$D_k \tilde{b}_j = D_j \tilde{b}_k\,, \quad k = 0, 1, \ldots, n\,,\ j = 1, 2, \ldots, n.$$

This is a non-linear first order hyperbolic system. Moreover we also have

$$da + [a, a] + \frac{1}{4}[\tilde{b}, \tilde{b}] \;=\; 0.$$

In other words, all in all we have

$$
\begin{aligned}
&\text{(a)} \quad \frac{\partial}{\partial t}\tilde{b}_0 - \sum \frac{\partial}{\partial x_j} + (a \cdot \tilde{b})_{\text{ space-time}} \;=\; 0\\
&\text{(b)} \quad d\tilde{b} + a \wedge \tilde{b} \;=\; 0\\
&\text{(c)} \quad da + [a,a] = R\,[\tilde{b},\tilde{b}]\\
&\text{(d)} \quad \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\, a_j = 0
\end{aligned}
$$

We refer to this as the *Gauged Wave Map* system.

Remarks.

1. Equation (c) is not determined by the rest of the data since the curvature depends on the original map (and gauge change). No general formula is available. This would not preclude *a priori* estimates. However, the estimates for our global existence and uniqueness theorem for wave maps are done in Besov spaces (which here prove inferior to the Lorentz spaces). The equation
$$da + [a,a] = R(x)[b,b]$$
however behaves 'badly' (for bounded $R(x)$) in this context. Hence we must restrict the manifold M to a group or a Riemannian symmetric space in which case $R(x) \equiv R$, a constant.

2. At this stage, (a)-(b) is very similar to the *modified Schrödinger map* system but not as nice to work with because of the presence of the $\frac{\partial a_0}{\partial t}$ terms. So we go one step further and convert it to a single equation using Hodge theory. From now on we abuse notation and call $\tilde{b}$ just b.

THEOREM. *Let $b = d\phi + d^*\psi$. Then the gauged wave map system can be rewritten as*

$$
\begin{aligned}
&\textit{(a)} \quad \Box\phi + (a,b) = 0\\
&\textit{(b)} \quad \Box\psi + a \wedge b = 0\\
&\textit{(c)} \quad b = d\phi + d^*\psi\\
&\textit{(d)} \quad da + [a,a] = R[b,b]\\
&\textit{(e)} \quad \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\, a_j = 0.
\end{aligned}
$$

The initial data on ϕ and ψ can be taken to be

$$
\begin{aligned}
&\phi(0,x) = 0\,, && \psi(0,x) = 0\\
&\frac{\partial\phi}{\partial t}(0,x) = b_0(0,x) && \frac{\partial\psi}{\partial t}_{0,j}(0,x) = b_j(0,x)\\
&\frac{\partial}{\partial t}\psi_{j,k}(0,x) = 0 \qquad j,\,k \neq 0
\end{aligned}
$$

Since the gauged wave map system is still overdetermined we proceed as in the Schrödinger case and consider a subset of it. Incorporating (e) into (d) we obtain the following.

COROLLARY. *Suppose $M = G$ or $M = H/G$. Then a subset of the gauged wave map equations (a)- -(e)has a structure of a non-linear wave system of integral differential operators.*

$$
\begin{aligned}
&(a) \quad \Box\phi + (a,b) = 0\\
&(b) \quad \Box\psi + a\wedge b = 0\\
&(c) \quad b = d\phi + d^*\psi\\
&(d) \quad \Delta a_j + \sum_{j=1}^{n} \frac{\partial}{\partial x^k}[a_k, a_j] + \frac{\partial}{\partial x^k}[b_k, b_j] = 0,\ j = 0, 1, \dots n
\end{aligned}
\tag{4}
$$

We refer to this system as the *Modified Wave Map* system - **MWM**

THEOREM (Well posedness of the MWM). *There exists $\varepsilon > 0$ such that whenever the initial data $\|(f,g)\|_{\dot H^{n/2}\times \dot H^{n/2-1}} < \varepsilon$, the system (4) has a unique global solution $v = (\phi, \psi)$ which belongs both to*

$$\bullet\, L^\infty(\mathbb{R}; \dot H^{n/2}_x) \cap L^2(\mathbb{R}; \dot B^1_{2n,2}) \qquad \textit{and}$$

$$\bullet\, W^{1,\infty}(\mathbb{R}; \dot H^{n/2-1}_x) \cap W^{1,2}(\mathbb{R}; \dot B^0_{2n,2}).$$

Moreover, there is stability; i.e.

$$\sup_t \|v_1 - v_2\|_{\dot H^{n/2}} \lesssim \|(f_1, g_1) - (f_2, g_2)\|_{\dot H^{n/2}\times \dot H^{n/2-1}}$$

provided the r.h.s. is small enough.

Remark. The theorem above gives the *existence* part of our Main Theorem on *wave maps*. The uniqueness of solutions to the MWM however is *solely* in the Besov spaces which is **not enough** to claim the solution to the MWM system came from a wave map. Thus an additional argument is needed. The following is a stronger uniqueness result which will indeed suffice to return to the wave map (ie. will give uniqueness of the *original* wave map).

THEOREM (Uniqueness). *Suppose $(v_1\,,\, a_1)$ and $(v_2\,,\, a_2)$ are two solutions to*

$$
\begin{aligned}
\Box v + B(a, dv) \ &= \ 0\\
\Delta a + divB(a,a) + divB(dv, dv) \ &= \ 0.
\end{aligned}
$$

such that $dv_j = b_j$, for $j = 1, 2$ are small in $L^\infty_t L^n_x$. Suppose that $dv_j = b_j \in L^2_t L^{2n}_x$ for $j = 1, 2$. Assume in addition that $a_1 = a_1(v_1) \in L^1_t L^\infty_x$. Then $v_1 \ = \ v_2$.

Remarks.

1. The smallness of dv_j in $L^\infty_t L^n_x$ is the necessary condition to solve the 'gauged' equation. Also note that it is not necessary for $a_2 = a_2(v_2) \in L^1_t L^\infty_x$.
2. The proof follows the scheme devised by Shatah-Struwe to establish uniqueness and essentially follows via energy estimates under minimal assumptions on the solution.

By differentiating the MWM system and observing that the resulting nonlinearity has the same bilinear structure -for which the main multiplication estimates hold- the following regularity result follows.

THEOREM (Higher Regularity). *Suppose the initial data (f,g) to (4.1) is in $H^{n/2+1}\times H^{n/2}$ and has sufficiently small $\dot H^{n/2}\times\dot H^{n/2-1}$ norm. Then the solution v to the Cauchy problem (4.1) with initial data (f,g) can be continued in $H^{n/2+1}\times H^{n/2}$ globally in time. Furthermore, we have the global bounds*

$$\|v\|_{L^\infty_t(\mathbb{R};\dot H^{n/2+1}_x)}\lesssim \|(f,g)\|_{\dot H^{n/2+1}_x\times\dot H^{n/2}_x}.$$

2.2 The Return to the Map.

The well-posedness results on the modified wave map apply to a larger class of formal solutions (a,b) to the equation than those which come from wave maps.

Our method of using the results on the modified wave map equation to show existence of wave maps is similar to the idea we used for non-linear Schrödinger and not very different from the technique used by Shatah-Struwe.

Roughly, first regularize the data to the wave map system. Then it has a *local smooth* solution. If *in addition* the $\dot H^{n/2}\times\dot H^{n/2-1}$ norm of the data is *sufficiently small* the local smooth solotion is *global and smooth* and satisfies the a priori global estimates satisfied by the solution to the MWM system. This a priori estimate is now used to pass to the limit.

The translation depends on the compactness of M (or certain bounds on the isometric Nash embedding of a non-compact M in an Euclidean space).

THEOREM (Local). *Let $s:[0,T]\times\mathbb{R}^n\to G$ be a wave map in a time interval $[0,T]$ such that $ds\in L^\infty_tL^n_x\cap L^2_tL^{2n}_x$. Assume the initial data is in $\dot H^{n/2}\times\dot H^{n/2-1}$ and has sufficiently small norm. Then s is a gauge transformation of a modified wave map, and $s\in\dot H^{n/2}\times\dot H^{n/2-1}$ remains small in $[0,T]$. Moreover, if the initial data is in $\dot H^{n/2+1}\times\dot H^{n/2}$, then $s\in L^\infty_t\dot H^{n/2+1}$ and $\frac{\partial s}{\partial t}\in L^\infty_t\dot H^{n/2}$ for the time the solution exists.*

COROLLARY. *Suppose $s:[0,T]\times\mathbb{R}^n\to G$ is a wave map with $ds\in L^\infty_tL^n_x\cap L^2_tL^{2n}_x$. Suppose, in addition, the data at any point of time is small in $\dot H^{n/2}\times\dot H^{n/2-1}$. Then s exists for all time and 'have the right estimates'.*

THEOREM (Global smooth). *If $(s,\nu)\in H^{n/2+1}\times H^{n/2}$ are initial data for a wave map and $(s,\nu)\in\dot H^{n/2}\times\dot H^{n/2-1}$ has small enough norm, then there exists a unique global solution with 'the right estimates'.*

THEOREM (Global). *Let $(s,\nu)\in\dot H^{n/2}\times\dot H^{n/2-1}$ be small data for a wave map into a compact group or symmetric space. Then there exists a unique global solution, which is a gauge transformation of a solution to the modified wave equation with 'the right estimates'.*

2.3 Well posedness of the MWM system.

In this section we will sketch the main ideas involved in the proof of our theorem above establishing the well posedness of the MWM system. The techniques follow those of Tao in [**46**].

Let $\phi(t,x)$ be a function on $\mathbb{R}\times\mathbb{R}^n$. The *spatial* Fourier transform $\hat{\phi}(t,\xi)$ is then given by

$$\hat{\phi}(t,x)=\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\phi(t,x)\,dx$$

For each integer k let $P_k(\phi)$ be the usual Littlewood-Paley projection operator onto spatial frequencies $|\xi|\lesssim 2^k$ and let the operator Q_k be the projection onto the spatial frequency annulus $|\xi|\sim 2^k$ given by the formula, $Q_k:=P_k-P_{k-1}$.

Let $n\geq 4$. Let $j=0$ or $j=1$ and $k\in\mathbb{Z}$. Following Tao [**46**] we introduce the *Strichartz space at frequency* 2^k, which we will denote by $S_k^{(-j)}(\mathbb{R}\times\mathbb{R}^n)$ to be the space of functions whose space-time norm is given by:

$$\|Q_k(\phi)\|_{S_k^{(-j)}}:=\sup_{q,r\in\mathcal{A}} 2^{k(\frac{1}{q}+\frac{n}{r}-j)}\big(\|Q_k(\phi)\|_{L_t^qL_x^r}+2^{-k}\|\partial_tQ_k(\phi)\|_{L_t^qL_x^r}\big),$$

where $\mathcal{A}:=\{(q,r)\,:\,2\leq q,\,r\leq\infty,\,\dfrac{1}{q}+\dfrac{n-1}{2r}\leq\dfrac{n-1}{4}\}$ is the set of wave admissible Strichartz exponents.When $j=0$ the spaces above are $\dot{H}^{n/2}$- normalized. For $j=1$ these are $\dot{H}^{n/2-1}$ -normalized.

Strichartz Estimates ([**20**] [**46**]. *Let $k\in\mathbb{Z}$ and let $Q_k(\phi)(t,x)$ be any function on $\mathbb{R}\times\mathbb{R}^n$ with spatial Fourier support on the annulus $|\xi|\sim 2^k$. Then*

$$\|Q_k(\phi)\|_{S_k^{(-j)}}\lesssim\|Q_k(\phi)(0,\cdot)\|_{\dot{H}_x^{n/2-j}}+\|\partial_tQ_k(\phi)(0,\cdot)\|_{\dot{H}_x^{n/2-j-1}}+ \\ +\,2^{k(\frac{n}{2}-(j+1))}\,\|\Box Q_k(\phi)\|_{L_t^1L_x^2}.$$

Definition.

Let $j=0,1$. We denote by $\mathcal{S}^{(-j)}$ the space of functions on $\mathbb{R}\times\mathbb{R}^n$ whose norm is given by

$$\|\phi\|_{\mathcal{S}^{(-j)}}:=\big(\sum_{k\in\mathbb{Z}}\|Q_k(\phi)\|^2_{S_k^{(-j)}}\big)^{1/2}.$$

Remark. Let f be a function on $\mathcal{S}^{(-j)}$. Then clearly,

$$\|\partial_x^{n/2-1}Q_k(f)\|_{L_t^\infty L_x^2}\sim 2^{k(n/2-j)}\|Q_k(f)\|_{L_t^\infty L_x^2}\leq\|Q_k(f)\|_{S_k^{(-j)}}.$$

Hence by taking ℓ^2 norms both sides

$$\big(\sum_{k\in\mathbb{Z}}2^{2k(n/2-j)}\|Q_k(f)\|^2_{L_t^\infty L_x^2}\big)^{1/2}\lesssim\|f\|_{\mathcal{S}^{(-1)}}$$

In other words, $\mathcal{S}^{(-j)}\hookrightarrow L_t^2\dot{H}_x^{n/2-j}$.

Multiplication and a priori estimates.

Definition. *Let us denote by $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{G}$ the following sets of pairs.*

$$\mathcal{C} =: \{(q,p) \in \mathcal{A} : \frac{1}{q} + \frac{n}{p} \leq 1^-\}$$

$$\mathcal{D} =: \{(q,p) : \frac{1}{2q} + \frac{n-1}{4p} \leq (\frac{n-1}{4})^-\}$$

$$\mathcal{E} =: \{(q,p) : \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}; \ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{with } (q_1,p_1) \in \mathcal{A} \text{ and } (q_2,p_2) \in \mathcal{C}\}$$

where $\mathcal{A}$ is the set of all (wave) admissible pairs. Finally, let $\mathcal{G} =: \mathcal{D} \cap \mathcal{E}$. Note that $\mathcal{A} \subset \mathcal{G}$.

Definition. *Let $\mathcal{S}_+^{(-1)}$ be the space of functions on $\mathbb{R} \times \mathbb{R}^n$ whose norm is given by*

$$\|\phi\|_{\mathcal{S}_+^{(-1)}} := \sum_{k\in\mathbb{Z}} \|Q_k(\phi)\|_{S_{k_+}^{(-1)}}$$

where

$$\|\Phi\|_{\mathcal{S}_{k_+}^{(-1)}} := \sup_{(q,p)\in\mathcal{G}} 2^{k\,(\frac{1}{q}+\frac{n}{p}-1)} \|\Phi\|_{L_t^q L_x^p} + \sup_{(q,p)\in\mathcal{G}, q\geq 2} 2^{k\,(\frac{1}{q}+\frac{n}{p}-1)} 2^{-k} \|\partial_t \Phi\|_{L_t^q L_x^p}$$

Remark. The set $\mathcal{G}$ is *a lot bigger* that the set $\mathcal{A}$ of Strichartz admissible pairs. In particular it contains the pairs $(2,2)$ and $(1,\infty)$ which are <u>not</u> in $\mathcal{A}$. This will be absolutely crucial in controlling the nonlinearity and closing the estimates; as we will see below.

Definition.
We denote by $\mathcal{B}_p$ be the Banach space of functions on $\mathbb{R} \times \mathbb{R}^n$ whose norm is given by

$$\|f\|_{\mathcal{B}_p} := \big(\sum_{k\in\mathbb{Z}} \|Q_k(f)\|^p_{L_t^1 L_x^\infty}\big)^{1/p}$$

for $1 \leq p < \infty$ and suitable modified with the ℓ^∞-norm when $p = \infty$

Lemma. *We have the following embeddings*

- $\mathcal{S}_+^{(-1)} \hookrightarrow \mathcal{S}^{(-1)}$
- $\mathcal{S}_+^{(-1)} \hookrightarrow \mathcal{B}_1 \hookrightarrow \mathcal{B}_p$
- $\mathcal{S}_+^{(-1)} \hookrightarrow L_t^q \dot{B}^s_{\tilde{p},2}$

for all $q \geq 2, \tilde{p} \geq 2$ and $s = \frac{1}{q} + \frac{n}{\tilde{p}} - 1$.

Notation. *Let $|\nabla|^{-1} = \nabla\Delta^{-1}$ be the pseudodifferential operator defined by*

$$\widehat{|\nabla|^{-1} f}(t,\xi) = \frac{1}{|\xi|}\hat{f}(t,\xi).$$

PROPOSITION (Main Multiplication Estimate) [**34**].

$$|\nabla|^{-1} : \mathcal{S}^{(-1)} \times \mathcal{S}^{(-1)} \longrightarrow \mathcal{S}^{(-1)}_{+}$$

THEOREM. *Let $a \in \mathcal{S}^{(-1)}_{+}$ and $b \in \mathcal{S}^{(-1)}$ then*

$$\Big(\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)} \|Q_k(a\cdot b)\|^2_{L^1_t L^2_x}\Big)^{1/2} \lesssim \|a\|_{\mathcal{S}^{(-1)}_{+}} \|b\|_{\mathcal{S}^{(-1)}}$$

SKETCH OF PROOF.

We start by performing a Littlewood-Paley decomposition of a and b. We obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)} \|Q_k(a\cdot b)\|^2_{L^1_t L^2_x} \lesssim \\
&\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)} \|\sum_{m>5} \big(Q_k(Q_k(a)\cdot Q_{k-m}(b))\big)\|^2_{L^1_t L^2_x} + \\
&\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)} \|\sum_{m>5} \big(Q_k(Q_{k-m}(a)\cdot Q_k(b))\big)\|^2_{L^1_t L^2_x} + \\
&\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)} \|\sum_{k<l} \big(Q_k(Q_l(a)\cdot Q_l(b))\big)\|^2_{L^1_t L^2_x}.
\end{aligned}$$

Observe that by the multiplication estimates a belongs to both $\mathcal{B}_1$ and $L^2_t \dot{B}^{n/2-1/2}_{2,2}$. To get an idea of the importance of the $L^1_t L^\infty_x$ estimate in a we estimate the second term where the high-low interactions take place. The philosophy from Strichartz is that then *large* r is good on the low frequency term.

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)} \|\sum_{m>5} \big(Q_k(Q_k(b)\cdot Q_{k-m}(a))\big)\|^2_{L^1_t L^2_x} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)} \|Q_k(b)\|^2_{L^\infty_t L^2_x} \Big(\sum_{m>5} \|Q_{k-m}(a)\|_{L^1_t L^\infty_x}\Big)^2 \\
&\lesssim \|a\|^2_{\mathcal{S}^{(-1)}_{+}} \sum_{k\in\mathbb{Z}} \|Q_k(b)\|^2_{\mathcal{S}^{(-1)}_k} \\
&\lesssim \|a\|^2_{\mathcal{S}^{(-1)}_{+}} \|b\|^2_{\mathcal{S}^{(-1)}}.
\end{aligned}$$

We are ready now to consider the Modified Wave Map system. Let us denote by $B(a,b)$ any finite linear combination of functions $a \in \mathcal{S}^{(-1)}_{+}$ and $b \in \mathcal{S}^{(-1)}$ of the form $\sum_{\kappa,\ell} c_{\kappa\ell} a_\kappa b_\ell$ where $a_\kappa \in \mathcal{S}^{(-1)}_{+}$, $b_\ell \in \mathcal{S}^{(-1)}$ and $c_{\kappa\ell} \in \mathbb{C}$.

We consider the MWM system of coupled wave equations in R^{n+1}, $n \geq 4$.

$$\begin{aligned}
\Box v &= B(a,b) \sim a\cdot b \\
v(x,0) &= f(x) \\
v_t(x,0) &= g(x)
\end{aligned}$$

(5)

where $v = (\varphi, \psi)$, $b = d\varphi + div_{(sp.t)}\,\psi$ and

$$da = [a,a] + [b,b], \quad i.e. \quad a = |\nabla|^{-1}[a,a] + |\nabla|^{-1}[b,b]$$

According to the Strichartz estimates we have

$$\|Q_k(v)\|_{\mathcal{S}_k} \lesssim \|f_k\|_{\dot{H}^{n/2}} + \|g_k\|_{\dot{H}^{n/2-1}} + \|Q_k(|\nabla|^{-1}[b,b]\cot b)\|_{L^1_t L^2_x}$$

More generally we have:

Lemma (A priori estimate). *Let $a \in \mathcal{S}_+^{(-1)}$ and $b \in \mathcal{S}^{(-1)}$. Then the solution to the MWM system (5) with initial data $(f,g) \in \dot{H}^{n/2} \times \dot{H}^{n/2-1}$ satisfies*

$$\|v\|_{\mathcal{S}} \lesssim \|f\|_{\dot{H}^{n/2}} + \|g\|_{\dot{H}^{n/2-1}} + \|a\|_{\mathcal{S}_+^{(-1)}}\|b\|_{\mathcal{S}^{(-1)}}$$

Proof. Let us denote by $v_k = Q_k(v)$. By the Strichartz's estimates, we have that

$$\|v_k\|_{\mathcal{S}_k} \lesssim \|f_k\|_{\dot{H}^{n/2}} + \|g_k\|_{\dot{H}^{n/2-1}} + 2^{k(n/2-1)}\|B(a,b)\|_{L^1_t L^2_x}$$

from where by the estimate on the nonlinearity we have that

$$\begin{aligned}
\|v\|_{\mathcal{S}} &= \big(\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathcal{S}_k}\big)^{1/2} \\
&\lesssim \|f\|_{\dot{H}^{n/2}} + \|g\|_{\dot{H}^{n/2-1}} + \big(\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}\|B(a,b)\|^2_{L^1_t L^2_x}\big)^{1/2} \\
&\lesssim \|f\|_{\dot{H}^{n/2}} + \|g\|_{\dot{H}^{n/2-1}} + \|a\|_{\mathcal{S}_+^{(-1)}}\|b\|_{\mathcal{S}^{(-1)}}
\end{aligned}$$

as desired.

Wrapping up.

The proof of the well posedness of the MWM system proceeds by Picard's iteration relying on the *a priori* estimates as well as the necessary smalless of the data.

Suppose $\|(f,g)\|_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}} = \delta$ and let v_0 be the solution to

$$\Box v_0 = 0; \qquad v_0(0,\cdot) = f \quad \partial_t v_0(0,\cdot) = g.$$

By the Strichartz's estimates

$$\|v_0\|_{\mathcal{S}} \leq c_1\|(f,g)\|_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}} = c_1\delta.$$

Now, $v_0 = (\varphi_0, \psi_0)$ produces $b_0 = d\varphi_0 + div_{(sp.t)}\,\psi_0$ with $\|b_0\|_{\mathcal{S}^{(-1)}} \leq c_2\|v_0\|_{\mathcal{S}} \leq c_3\delta$.

Next, the multiplication estimates allow one to perform a fixed point argument to produce a_0 from b_0 by solving

$$a_0 = |\nabla|^{-1}[a_0,a_0] + |\nabla|^{-1}[b_0,b_0].$$

Moreover, $\|a_0\|_{\mathcal{S}_+^{(-1)}} \leq c_4\|b_0\|^2_{\mathcal{S}^{(-1)}} \leq c_5\delta^2$

Let v_1 be the solution of

$$\Box v_1 = B(a_0,b_0) \qquad v_1(0,\cdot) = f \quad \partial_t v_1(0,\cdot) = g.$$

By the *a priori* estimate,

$$\|v_1\|_{\mathcal{S}} \leq c_0\big(\delta + \|a_0\|_{\mathcal{S}_+^{(-1)}}\|b_0\|_{\mathcal{S}^{(-1)}}\big) \leq 2c_0\delta$$

provided δ is small enough.

We proceed next by induction to show that for any $j \geq 0$, $\|b_j\|_{\mathcal{S}} \leq 2c_2c_o\delta$, $\|a_j\|_{\mathcal{S}} \leq c_5\delta^2$ and thus $\|v_{j+1}\|_{\mathcal{S}} \leq 2c_0\delta$ provided $\delta > 0$ is small enough (indep. of j), where v_{j+1} is the solution to

$$\Box v_{j+1} = B(a_j, b_j) \qquad v_{j+1}(0,\cdot) = f \quad \partial_t v_{j+1}(0,\cdot) = g.$$

Note that once again by the *a priori* estimates

$$\|v_{j+1}\|_{\mathcal{S}} \leq c_0\big(\|(f,g)\|_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}} + \|a_j\|_{\mathcal{S}_+^{(-1)}}\|b_j\|_{\mathcal{S}^{(-1)}}\big).$$

Lastly, for the differences, we prove that

$$\begin{aligned}\|v_{j+2} - v_{j+1}\|_{\mathcal{S}} &\leq c\big(\|a_{j+1} - a_j\|_{\mathcal{S}_+^{(-1)}}\|b_{j+1}\|_{\mathcal{S}^{(-1)}} \\ &\quad + \|a_{j+1}\|_{\mathcal{S}^{(-1)}}\|b_{j+1} - b_j\|_{\mathcal{S}^{(-1)}}\big) \\ &\leq c\delta^2\|v_{j+1} - v_j\|_{\mathcal{S}}\end{aligned}$$

Finally, by choosing δ small enough we have that

$$\|v_{j+2} - v_{j+1}\|_{\mathcal{S}} \leq \frac{1}{2}\|v_{j+1} - v_j\|_{\mathcal{S}}.$$

Hence v_j is Cauchy in $\mathcal{S}$, thus establishing existence and uniqueness. For the stability result one proceeds in the same fashion as in the proof of being Cauchy; thus concluding the proof of the theorem.

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