Math 132 Exam 3 Solutions 4/23/09

1. (a) [8%] $0.\overline{025} = \frac{25}{10^3} + \frac{25}{10^6} + \frac{25}{10^9} + \cdots = \boxed{\sum_{n=1}^{\infty} \frac{25}{10^{3n}}}$ or $0.\overline{025} = \frac{25}{1000} + \frac{25}{1000^2} + \frac{25}{1000^3} + \cdots = \boxed{\sum_{n=1}^{\infty} \frac{25}{1000^n}}$

   Note: Another, correct, way to write the answer is $\sum_{n=1}^{\infty} \frac{25}{1000} \left(\frac{1}{1000}\right)^{n-1}$.

   However, $\sum_{n=0}^{\infty} \frac{25}{1000^n}$ and $\sum_{n=1}^{\infty} \frac{25}{1000} \left(\frac{1}{1000}\right)^{n}$ are very *wrong!*

   (b) [8%] This is a *geometric series* having initial term $a = \frac{25}{1000}$ and ratio $r = \frac{1}{1000}$. Hence:

$$\sum_{n=1}^{\infty} \frac{25}{1000^n} = \frac{a}{1-r} = \frac{25/1000}{1 - 1/1000} = \frac{25}{1000-1} = \boxed{\frac{25}{999}}$$

2. (a) [8%] By definition, the sum of the series is the limit of its sequence of partial sums:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3^n - 9}{3^{n+1}} \\ &= \lim_{n\to\infty} \left(\frac{3^n}{3^{n+1}} - \frac{9}{3^{n+1}}\right) = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{9}{3^{n+1}}\right) = \frac{1}{3} - 0 = \boxed{\frac{1}{3}} \end{aligned}$$

   Notes: s_n is the nth partial sum and *not* the nth term a_n — so the series is *not* $\sum_{n=1}^{\infty} \frac{3^n-9}{3^{n+1}}$! For (a) it's pointless to find a formula for a_n; even if you do (using $a_n = s_n - s_{n-1}$), the fact that then you find $\lim_{n\to\infty} a_n = 0$ does *not* tell you that $\sum_{n=1}^{\infty} a_n = 0$ converges!

   (b) [8%] For each $n = 2, 3, \ldots,$

$$\begin{aligned} a_n = s_n - s_{n-1} &= \frac{3^n - 9}{3^{n+1}} - \frac{3^{n-1} - 9}{3^{(n-1)+1}} \\ &= \frac{3^n - 9}{3^{n+1}} - \frac{3^{n-1} - 9}{3^n} = \frac{3^n - 9 - 3(3^{n-1} - 9)}{3^{n+1}} = \frac{3^n - 9 - 3^n + 27}{3^{n+1}} = \frac{18}{3^{n+1}} \qquad \text{(OK, or simplify more:)} \\ &= \frac{3^2 \cdot 2}{3^2 \cdot 3^{n-1}} = \boxed{\frac{2}{3^{n-1}}} \end{aligned}$$

   And: $a_1 = s_1 = \frac{3^1 - 9}{3^{1+1}} = \frac{-6}{3^2} = -\frac{2}{3}$

3. (a) [8%]

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}/(n+1)}{(x-3)^n/n}\right| &= \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{(x-3)^n} \cdot \frac{n}{n+1}\right| = \lim_{n\to\infty} \left|(x-3) \cdot \frac{n}{n+1}\right| \\ &= |x-3| \lim_{n\to\infty} \frac{n}{n+1} = |x-3| \cdot 1 = |x-3| \end{aligned}$$

   According to the Ratio Test, then the power series converges when $|x-3| < 1$ and diverges when $|x-3| > 1$. Hence its radius of convergence is $\boxed{R = 1}$.

   (b) [8%] It remains to test the endpoints—where $|x-3| = 1$, in other words, where $x - 3 = \pm 1$.
   [*Note:* From (a), we already know that the power series converges when $|x-3| < 1$, that is, when $-1 < x - 3 < 1$, that is, when $2 < x < 4$; and that it diverges when $x < 2$ or $x > 4$.]

   Test $x - 3 = 1$ (that is, $x = 4$). The series is $\sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{1=n}^{\infty} \frac{1}{n}$, the **divergent** *harmonic series.*

   Test $x - 3 = -1$ (that is, $x = 2$). The series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the series whose terms are the *negatives* of the corresponding terms of the alternating harmonic series. Since the *alternating harmonic series* converges, this series **converges**, too.

   Conclusion: The series converges only for $-1 \le x - 3 < 1$, that is, $2 \le x < 4$. In other words, the series has $\boxed{\text{interval of convergence } [2, 4).}$

4. (a) [8%] The first term is $1/(-1) < 0$, but all other terms are positive because, for $n \geq 2$, $n^3 + 7n^2 - 9 \geq 0$. For $n \geq 2$,

$$\frac{1}{n^3 + 7n^2 - 9} \leq \frac{1}{n^3} \iff n^3 + 7n^2 - 9 \geq n^3 \iff 7n^2 \geq 9 \iff n^2 \geq \frac{9}{7},$$

and certainly $n^2 \geq \dfrac{9}{7}$ when $n \geq 2$. Hence indeed $\dfrac{1}{n^3 + 7n^2 - 9} \leq \dfrac{1}{n^3}$ for all $n \geq 2$.

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since it is a p-series with $p = 3 > 1$. Hence also $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3}$ converges.

By the **Comparison Test**, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 + 7n^2 - 9}$ also converges. So the given series $\boxed{\textbf{converges}}$.

(b) [8%]

$$\lim_{n \to \infty} \frac{\sqrt{n^2 + 7}}{2n + 3} = \lim_{n \to \infty} \frac{(1/n)\sqrt{n^2 + 7}}{(1/n)(2n + 3)} = \lim_{n \to \infty} \frac{\sqrt{(n^2 + 7)/n^2}}{2 + 3/n} = \lim_{n \to \infty} \frac{\sqrt{1 + 7/n^2}}{2 + 3/n} = \frac{1}{3} \neq 0$$

By the **Test for Divergence** (a.k.a. the ***n*th Term Test**), the given series $\boxed{\textbf{diverges}}$.

(c) [8%] The series is alternating, and the sequence of absolute values $b_n = \dfrac{1}{\ln n}$ of absolute values of its terms clearly has properties:

- $(b_n)_{n=1}^{\infty}$ is decreasing; and
- $\displaystyle\lim_{n \to \infty} b_n = 0$.

By the **Alternating Series Test** the given series $\boxed{\textbf{converges}}$.

5. (a) [8%] *Method 1: Use summation notation.* From $\dfrac{1}{1-r} = \displaystyle\sum_{n=0}^{\infty} r^n$ get

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n\, x^{2n}.$$

Then

$$f(x) = \frac{x}{1-4x^2} = x\,\frac{1}{1-4x^2} = x\sum_{n=0}^{\infty} 4^n\, x^{2n} = \sum_{n=0}^{\infty} x \cdot 4^n\, x^{2n} = \sum_{n=0}^{\infty} 4^n\, x^{2n+1}$$

Note: An answer like $\displaystyle\sum_{n=1}^{\infty} 4^n\, x^{2n+1}$ is *wrong:* it's missing the constant term.

Method 2: Avoid summation notation. From $\dfrac{1}{1-r} = 1 + r + r^2 + r^3 + r^4 + \cdots$ get

$$\begin{aligned}\frac{1}{1-4x^2} &== 1 + (4x^2) + (4x^2)^2 + (4x^2)^3 + +(4x^2)^4 + \cdots \\ &= 1 + 4x^2 + 16x^4 + 64x^6 + 256x^8 + \cdots .\end{aligned}$$

Then:

$$\begin{aligned}f(x) = x\,\frac{1}{1-4\,x^2} &= x\,\left(1 + 4x^2 + 16x^4 + 64x^6 + 256x^8 + \cdots\right) \\ &= \boxed{x + 4x^3 + 16x^5 + 64x^7 + 256x^9 + \cdots}\end{aligned}$$

(b) [8%] The series $\dfrac{1}{1-4x^2}$ converges if and only if $|4x^2| < 1$. Now

$$|4x^2| < 1 \iff |x|^2 < \frac{1}{4} \iff |x| < \frac{1}{2}$$

Since multiplying a series be a given number does not change whether it converges, the power series found in (a) also converges if and only if $\boxed{|x| < \frac{1}{2}}$, in other words, when $-\frac{1}{2} < x < \frac{1}{2}$.

6. (a) **[4%]** The fifth partial sum S_5 is:

$$S_5 = \sum_{n=1}^{5} (-1)^{n-1} \frac{1}{n^2+1} = \frac{1}{1^2+1} - \frac{1}{2^2+1} + \frac{1}{3^2+1} - \frac{1}{4^2+1} + \frac{1}{5^2+1}$$
$$= \frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \frac{1}{26} = \frac{839}{2210} \approx \boxed{0.379638}$$

(b) **[8%]** This is an alternating series. The sequence of absolute values $b_n = \dfrac{1}{n^2+1}$ of its terms satisfies the conditions:

- $(b_n)_{n=1}^{\infty}$ is decreasing; and
- $\lim\limits_{n\to\infty} b_n = 0$.

Hence the theory related to the Alternating Series Test applies: the error R_5 of the approximation satisfies

$$\left|R_5\right| \leq \left|\frac{(-1)^{(5+1)+1}}{(5+1)^2+1}\right| = \frac{1}{37} = 0.\overline{027} \leq \boxed{0.027028}$$

Note: It would be wrong to conclude $|R_5| \leq 0.027027$, since the upper bound obtained is the repeating decimal $0.027027027\ldots$, which is larger than 0.027027.