Algebra 411.2 Homework 8

Due November 10, in class.
All answers should be justified.

♡

0. Read chapters 10 and 11 in the book. (This is largely new material so one reads it to prepare for the lectures.)

♡

Left and right cosets of a subgroup. Let H be a subgroup of a group $(G, \cdot)$. For any element $g \in G$ the *left coset* of H through g is the subset $gH \stackrel{\text{def}}{=} \{g{\cdot}h;\ h \in H\}$. Similarly, the *right coset* of H through g is the subset $Hg \stackrel{\text{def}}{=} \{h{\cdot}g;\ h \in H\}$.

Denote by G/H the set of all left cosets of H in G and by $H\backslash G$ the set of all right cosets of H in G.

1. (a) Show that if the group G is abelian then the left cosets and the right cosets are always the same: $gH = H$ for any $g \in G$.

(b) For the subgroup $N\mathbb{Z}$ of a $\mathbb{Z}$ find all cosets. How many cosets are there, i.e., howmany elements in $\mathbb{Z}/N\mathbb{Z}$?

2. Let H be a subgroup of a group $(G, \cdot)$.

(a) Show that for $g \in G$ the left multiplication function $L_g :\ H \to gH$ by $L_g(h) \stackrel{\text{def}}{=} g{\cdot}h$ is a well defined bijection between H and its left coset gH.

(b) Show that if H is finite then all left cosets $uH,\ u \in G$ of H and all right cosets $Hv,\ v \in G$ of H have the same number of elements: $|uH| = |Hv|$.

3. Let H be a subgroup of a group $(G, \cdot)$.

(a) Show that for any $u, v \in G$ the left costes uH and vH are either the same or disjoint. (In other words if uH and vH have some common element x then $uH = vH$).

♡

Normal subgroups. A subgroup N of a group G is said to be *normal* if it is preserved under conjugation by elements of G, i.e., for any $n \in N$ and any $g \in G$ the g-conjugate ${}^{g}n \stackrel{\text{def}}{=} gng^{-1}$ of n, is again in N.

4. Let $G = GL_2(\mathbb{R})$ and consider its subgroups

$$B \;=\; \{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix};\;\; a, b, x \in \mathbb{R},\; a \neq 0,\; b \neq 0\} \quad \text{and} \quad N \;=\; \{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix};\;\; y \in \mathbb{R}\}.$$

(a) Show that N is a normal subgroup of B.

(b) Show that N is not a normal subgroup of G.

(c) Find an element g in G such that the left coset gN and the right coset Ng are not the same.

5. (a) For any homomorphism of groups $\phi : (G, \cdot) \to (h, \cdot)$ show that $\mathrm{Ker}(\phi)$ is a normal subgroup of H.

(b) Find an example of a homomorphism of groups $\phi : (G, \cdot) \to (h, \cdot)$ such that $Im(\phi)$ is not a normal subgroup of H.

6. Problem 1 in section 11 in the book.

7. Problem 3 in section 11 in the book.

Remark. The last time to ask questions is Tuesday!