**Algebra 411.2 Homework 7**

*Due November $3^{rd}$, in class.*

**All answers should be justified.**

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**0.** Read chapters 8 and 9 in the book. (This is largely new material so one reads it to prepare for the lectures.)

In the notes read sections 1.2, 1.4, 4.4 and the appendix A.

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**1.** An element $g$ of a group $(G, \cdot)$ defines three functions from $G$ to $G$

- The left multiplication function $L_g : G \to G$ defined by $L_g(x) \stackrel{\text{def}}{=} g{\cdot}x$;
- The right multiplication function $R_g : G \to G$ defined by $R_g(x) \stackrel{\text{def}}{=} x{\cdot}g^{-1}$;
- The conjugation function $C_g : G \to G$ defined by $C_g(x) = g{\cdot}xg^{-1}$.

Prove that

(1) For $a \in G$

$$L_a \circ R_a = C_a = R_a \circ L_a.$$

(2) For $a, b \in G$ one has

$$L_a \circ L_b = L_{ab} \quad \text{and} \quad R_a \circ R_b = R_{ab} \quad \text{and} \quad C_a \circ C_b = C_{ab}.$$

Also, for the neutral element $e \in G$

$$L_e = R_e = C_e = id_G.$$

(3) For $a \in G$, the conjugation function $C_a : G \to G$ is an isomorphism of groups.

**2.** Let $p$ and $q$ be positive integers and $n = pq$.

(a) Show that the function $\phi : \mathbb{Z}_n \to \mathbb{Z}_p$ defined by: $\phi(k) = qk \mod p$, for $k \in \mathbb{Z}_n$; is a well defined homomorphism.

(b) Find a generator of the image subgroup $\mathrm{Im}(\phi) \subseteq \mathbb{Z}_p$.

(b) Find a generator of the kernel $\mathrm{Ker}(\phi) \subseteq \mathbb{Z}_n$.

[Recall what image and kernel are from Homework 56.] [If needed consider the example when $p = 2$ and $q = 3$.]

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**Actions of groups on sets.** *An action $*$ of a group $G$ on a set $X$ is a rule which assigns to each $g \in G$ and $x \in X$ an element $g * x$ of $X$; provided that the following properties are satisfied*

1. *(Associativity) For $a, b \in G$ and $x \in X$,* $\quad a * (b * x) = \ (ab) * x.$
2. *For $x \in X$,* $\quad e * x = \ x.$

**3.** ["Any group $G$ acts on itself."] Show that any group $G$ acts on the set $G$ by the conjugation rule

$$g * x \stackrel{\text{def}}{=} gxg^{-1} \text{ for } g \in G \quad \text{and} \quad x \in X = G.$$

[This is called the *conjugation action* of $G$ on itself.]

**Orbits of group actions.** *When a group $G$ acts on a set $X$ then the* orbit $G * a$ *of an element $a \in X$ is the subset of $X$*

$$G * a \stackrel{def}{=} \{g * a;\ g \in G\}.$$

*So, it consists of all elements $x$ of the set $X$ that are of the form $g * a$, i.e., such that $x$ can be obtained from $a$ by acting on $a$ by some element $g$ of $G$. The set of all orbits of $G$ in $X$ is denoted*

$$G \backslash X \stackrel{def}{=} \{G * a;\ a \in X\}.$$

*It is called the quotient of $X$ by the action of the group $G$.*

4. Consider the action of the group $G = S_3$ on itself by conjugation. Find all orbits of this action. (The orbits of $G$ on itself are called *conjugacy classes.*)

5. For the subgroup $H$ of a groups $(G, \cdot)$ show that

(1) The rule that associates to a pair of $h \in H$ and $g \in G$ the element $h * g \stackrel{\text{def}}{=} h \cdot g$ is an action of $H$ on $G$.
(2) The rule that associates to a pair of $h \in H$ and $g \in G$ the element $h \star g \stackrel{\text{def}}{=} g \cdot h^{-1}$ is an action of $H$ on $G$.
(3) In the case when $G = \mathbb{Z}$ and $H$ is the subgroup $3\mathbb{Z}$ of multiples of 3, find all orbits of the action $*$ of $H$ on $G$. How many of them are there?