Algebra 411.1

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Homework 5

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Due Wednesday April 8, in class.

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1. (a) For rings S_1 and S_2 consider the maps

$$S_1 \xrightarrow{i_1} S_1 \times S_2 \xleftarrow{i_2} S_2, \quad i_1(u) \overset{\text{def}}{=} (u, 1_{S_2}) \quad \text{and} \quad i_2(v) \overset{\text{def}}{=} (1_{S_1}, v)$$

and

$$S_1 \xleftarrow{p_1} S_1 \times S_2 \xrightarrow{p_2} S_2, \quad p_1(u,v) \overset{\text{def}}{=} u \quad \text{and} \quad p_2(u,v) \overset{\text{def}}{=} v.$$

Show that

(1) p_1, p_2 are morphisms of rings.
(2) i_1, i_2 are "morphisms of rings without unity", i.e., they preserve addition and mutilpication but they do not preserve units.

(b) Show that for any ring R the map

$$\iota: \ \text{Hom}(R, S_1 \times S_2) \ \longrightarrow \ \text{Hom}(R, S_1) \times \text{Hom}(R, S_2), \quad \iota(f) \overset{\text{def}}{=} \ (p_1 \circ f, p_2 \circ f);$$

is well defined and it is a bijection.

2. Let m, n be positive integers. Show that

(a) There is precisely one homorphism of rings

$$\phi: \ \mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_n.$$

(b) If m, n are relatively prime then ϕ is an isomorphism.

(c) If m, n are not relatively prime then the rings $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are not isomorphic.

3. Show that in the following pairs, the two rings are not isomorphic

(1) $\mathbb{R}$ and $\mathbb{C}$,
(2) $\mathbb{Z}$ and $\mathbb{R}$.

4. For the polynomial $P = X^3 + 10X^2 + 6X + 1 \ \in \ \mathbb{Z}[X]$, show that $P(x) = 0$ has no solutions in $\mathbb{Z}$.

5. Show that for rings R and S

(1) If $I \subseteq R$ and $J \subseteq S$ are ideaqls show that $I \times J = \ \{(x, y); \ x \in I \quad \text{and} \quad y \in J\}$ is an ideal in $R \times S$.
(2) Any ideal K in $R \times S$ is equal to $I \times J$ for some ideals $I \subseteq R$ and $J \subseteq S$.