Algebra 411.2

Homework 3

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All answers should be justified.

Due Wednesday February 25, in class.

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1. Let R be a ring. We say that a function $f: R \to R$ is an *automorphism* of a ring R if it f is an isomorphism of rings.

For any invertible element $a \in R^*$, we call the function

$$C_a: R \to R, \quad C_a(x) \stackrel{\text{def}}{=} axa^{-1}, \quad x \in R;$$

the *conjugation* by a. Prove that

- (a) $C_1 = id_R$.
- (b) For $a, b \in R^*$, $C_b \circ C_a = C_{ba}$.
- (c) For any $a \in R^*$, conjugation C_a is an automorphism of the ring R.
- (d) Conjugation C is an action of the group G on the set R.

[We also denote $C_a(x)$ by ${}^a x$. Then property (b) says that ${}^b({}^a x) = {}^{ba}x$.]

2. [Examples of conjugation.]

- (a) For $R = M_2(\mathbb{R})$ calculate
  (1) ${}^a\beta$ for $a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$.
  (2) ${}^s\beta$ and ${}^s\alpha$ for $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- (b) For $R = \mathbb{H}$:
  (1) Calculate conjugates[1]
  $$ {}^I J, \ {}^I K, \ J^K, \ {}^J I, \ {}^K I, \ {}^K J. $$
  (2) Use this to explain that
  $$ Z_{\mathbb{H}}(I) = span_{\mathbb{R}}\{1, I\} \text{ implies } Z_{\mathbb{H}}(J) = span_{\mathbb{R}}\{1, J\}. $$

[1] Use the multiplication table of I, J, K, not the matrix computations.

3. For a ring R let $\mathrm{Aut}(R)$ be the set of all automorphisms of the ring R. Prove that

- (a) $\mathrm{Aut}(R) \ni id_R$.
- (b) If $\alpha, \beta \in \mathrm{Aut}(R)$ then $\alpha \circ \beta \in \mathrm{Aut}(R)$.
- (c) If $\alpha \in \mathrm{Aut}(R)$ then α has inverse function $\alpha : R \to R$ and $\alpha^{-1} \in \mathrm{Aut}(R)$.
- (d) $\mathrm{Aut}(R)$ is a group.

4. ["Trivial" examples of automorphism groups.]

- (a) If $f : R \to S$ is an isomorphism of rings show that
  (1) For any subset $A \subseteq R$, $Z_S f(A) = f(Z_R(A))$.
  (2) An element $e \in R$ is called idempotent (or projector) if $e^2 = e$. We denote by $\mathcal{P}_R$ the set of all idempotents in R. Show that $f(\mathcal{P}_R) \subseteq \mathcal{P}_S$, and also that $f(\mathcal{P}_R) = \mathcal{P}_S$.
- (b) Show that the following automorphism groups are trivial:
  (1) $\mathrm{Aut}(\mathbb{Z})$.
  (2) $\mathrm{Aut}(\mathbb{Z}_n)$.
  (3) $\mathrm{Aut}(\mathbb{Q})$.

5. [A non-trivial example.]

- Determine the set $\mathcal{P}_{\mathbb{Z}\oplus\mathbb{Z}}$ of idempotents in $\mathbb{Z}\oplus\mathbb{Z}$.
- (b) How large is the group $\mathrm{Aut}(\mathbb{Z}\times\mathbb{Z})$?

6. Show that for any ring R, the conjugation construction

$$C : R^* \longrightarrow \mathrm{Aut}(R), a \mapsto C_a$$

is a morphism of groups.