Algebra 412

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Homework 2

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Due Wednesday Feb 18, in class.

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1. A subset I of a ring R is said to be an *ideal* if

(1) it is a subgroup of $(R, +)$ (i.e., it is nonempty and if $x, y \in I$ then $x - y \in I$), and
(2) it is closed under multiplication with elements of R, i.e.,

$$x \in I \textit{ and } a \in R \quad \Rightarrow \quad ax, xa \in I.$$

Prove that the set of cosets $R/I = \{r + I;\ r \in R\}$ is then a ring for the operations

$$(a+I) +_{R/I} (b+I) \overset{\text{def}}{=} (a+b) + I \quad \text{and} \quad (a+I) \cdot_{R/I} (b+I) \overset{\text{def}}{=} ab + I.$$

2. (a) Show that the following 2x2 matrices with complex entries

$$1 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad I \overset{\text{def}}{=} \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad J \overset{\text{def}}{=} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad K \overset{\text{def}}{=} \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right);$$

satisfy

(1) $1^2 = 1$ and $I^2 = J^2 = K^2 = -1$,
(2) $1 \cdot X = X = X \cdot 1$ for $X = 1, I, J, K$,
(3) $I \cdot J = K$, $J \cdot K = I$, $K \cdot I = J$ and $J \cdot I = -K$, $K \cdot J = -I$, $I \cdot K = -J$.

[Hint: equations involving 1 are obvious, check first that $I^2 = J^2 = K^2 = -1$ and $IJ = K$ by calculating with matrices. Then show that all the remaining relations follow from these.]

(b) Show that the subset of $M_2(\mathbb{C})$ defined by

$$\mathbb{H} \overset{\text{def}}{=} span_{\mathbb{R}}\{1, I, J, K\} = \{a1 + bI + cJ + dK;\ a, b, c, d \in \mathbb{R}\}$$

is a subring.

3. (a) The *centralizer* of a subset A of a ring R is the set $Z_A(R)$ consisting of all $r \in R$ which commute with all elements of A :

$$(\forall a \in A) \quad ar \;=\; ra.$$

Show that for any subset $A \subseteq R$, its centralizer $Z_A(R)$ is a subring.

(b) The center of a ring R is the subset $Z(R)$ consisting of all $a \in R$ which commute with all elements of R. Show that the center $Z(R)$ is a subring.

(c) For the ring $\mathbb{H}$ of quaternions determine the center $Z(\mathbb{H})$ and the centralizer $Z_I(\mathbb{H})$ of the element I of $\mathbb{H}$.

4. (a) Write the definition of "V is a vector space over $\mathbb{R}$".

(b) Write the definition of a basis of a vector space V over $\mathbb{R}$.

(c) Show that the multiplication of elements of $\mathbb{H}$ by elements of $\mathbb{R}$

$$r\cdot(a1 + bI + cJ + dK) \overset{\text{def}}{=} (ra)1 + (rb)I + (rc)J + (rd)K \quad \text{for} \quad a, b, c, d, r \;\in \mathbb{R};$$

makes $\mathbb{H}$ into a vector space.

(d) Show that $1, I, J, K$ is a basis of the vector space $\mathbb{H}$ over $\mathbb{R}$.

[Hint: if you do not remember (a) and (b) from Linear Algebra, look for the definition in the book.]

5. (a) Denote by $\mathbb{H} \ni x \mapsto \overline{x} \in \mathbb{H}$ the function that sends $x = a1 + bI + cJ + dK$ to $\overline{x} = a1 - bI - cJ - dK$. Show that it is an *antihomomorphism of rings*, i.e., that

$$\overline{x+y} \;=\; \overline{x} + \overline{y} \quad \text{and} \quad \overline{x\cdot y} \;=\; \overline{y}\cdot\overline{x} \quad \text{for} \;\; x, y \in \mathbb{H}.$$

This operation is called *conjugation of quaternions.*

(b) Define the *norm* of $x = a1 + bI + cJ + dK \in \mathbb{H}$ to be

$$|x| \overset{\text{def}}{=} \sqrt{a^2 + b^2 + c^2 + d^2} \;\in \mathbb{R}_{\geq 0}.$$

Show that

(1) $x\cdot\overline{x} = \overline{x}\cdot x = |x|^2\cdot 1$.
(2) $|x\cdot y| \;=\; |x|\cdot\,|y|, \quad x, y \in \mathbb{H}$.
(3) If $x \in \mathbb{H}$ is not zero, then the quaternion $\frac{1}{|x|^2}\cdot\overline{x}$ is the inverse of x. (So, $\mathbb{H}^* = \mathbb{H} - \{0\}$.)

[Hint: It is preferable to prove (3) using (1) then by computing with the formula $x = a1 + bI + cJ + dK$, since you will otherwise need to duplicate the work you did for (1).]