Algebra 412, Spring 09, The 1^{st} Sample Exam

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This is also Homework 4, due at the time of the exam.

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Do 5 of the following 8 problems.

Remember:

(1) All claims should be justified.
(2) Write the solutions in
- **complete**
- **legible sentences so that**
- **anybody**
- **can follow your reasoning!**

1. Find all idempotents in $\mathbb{Z}_{30}$.

2. Find all zero divisors in $\mathbb{Z}_{30}$.

3. Let F be a field such that $0 \neq 1$.

(a) Show that the $x^2 = 1$ iff $x^{-1} = x$.

(b) Show that there are precisely two solutions of $x^2 = 1$ in F.

(c) Show that if F is finite then the product of all nonzero elements is -1.

4. An element x of a ring R is said to be *nilpotent* if there is some positive integer n such that $x^n = 0$.

(a) Find all nilpotent elements in $\mathbb{Z}_{16}$ and in $\mathbb{Z}_{24}$.

(b) Show that if A is an integral domain then the only nilpotent is 0.

(c) Show that if x is nilpotent and $x^n = 0$ for $n > 0$, then $1 + x + x^2 + x^3 + \cdots + x^{n-1}$ is the inverse of $1 - x$.

5. Let e be an idempotent in a ring R. Show that

(1) $f = 1 - e$ is also an idempotent and $ef = 0 = fe$.
(2) $eR \stackrel{\text{def}}{=} \{ex;\ x \in R\}$ equals $\{y \in R;\ ey = y\}$.
(3) If R is commutative then eR has a natural structure of a ring (but with a different unit than R).
(4) If R is commutative then $\phi :\ eR \times fR \longrightarrow R$ defined by

$$\phi(x, y) \stackrel{\text{def}}{=}\ x + y, \quad x \in eR,\ y \in f;$$

is a homomorphism of rings.

6. Let $\mathrm{Hom}(R, S)$ be the set of all homomorphisms from the ring R to the ring S.[1]

(a) Find $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_3)$.

(b) Find $\mathrm{Hom}(\mathbb{Z}_{10}, \mathbb{Z}_6)$.

(c) Show that for any ring R there is precisely one homomorphism ϕ from $\mathbb{Z}$ to R.

7. (a) Prove that if $f : R \longrightarrow S$ is a homomorphism of rings then the *kernel of* f

$$\mathrm{Ker}(f) \stackrel{\text{def}}{=}\ \{r \in R;\ f(r) = 0\}$$

is an ideal in R.

(b) Prove that the function

$$\phi :\ R/I \longrightarrow S, \quad \phi(r + I) \stackrel{\text{def}}{=}\ f(r), \quad r \in R$$

is

(1) well defined,
(2) a homomorphism.

8. Let R be a ring. Show that

(a) For any ideals I, J in R, the subset $I \cap J \subseteq R$ is an ideal.

(b) For any ideals I, J in R, the subset

$$I + J\ \stackrel{\text{def}}{=}\ \{x + y;\ x \in I \quad \text{and} \quad y \in J\} \subseteq R\ ;$$

is an ideal in R.

[1] One also says "morphism" instead of "homomorphism".