Algebra 411.2

EXAM 3

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All answers should be justified.

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Do any 6 problems, For each problem except problem 6, you can assume that the claims of preceding problems are known (regardless of whether you have solved previous problems).

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The exam is due the last day of the exam period.

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1. Let I be an ideal in a ring R and denote by $q:\ R \to R/I$ the canonical quotient map

$$q(r) = \ r + I, \quad r \in R.$$

(a) Let J be an ideal in R which contains I, i.e., $J \supseteq I$. Prove that the quotient group J/I is an ideal in the quotient ring R/I. [1] (b) For any subset K of R/I, we define its "pull back to R" to be the subset of R consisting of all $r \in R$ which are sent to K by the map q. The notation for this pull back is $q^{-1}(K)$, so

$$q^{-1}(K) \ \stackrel{\text{def}}{=} \ \{r \in R;\ q(R) \in K\} \ = \ \{r \in R;\ r + I \in K\} \quad \subseteq \quad R.$$

Prove that if $K \subseteq R/I$ is an ideal in R/I then its pull back $q^{-1}(K)$ is an ideal in R and it contains I.

(c) Denote by $\mathcal{J}$ the set of all ideals J in R that contain I. Denote by $\mathcal{K}$ the set of all ideals K in R/I. Then (a) and (b) define two procedures of passing between $\mathcal{J}$ and $\mathcal{K}$, i.e., two functions

(1) $\mathcal{A}:\ \mathcal{J} \to \mathcal{K}$ by $\mathcal{A}(J) = \ J/I$, and
(2) $\mathcal{B}:\ \mathcal{K} \to \mathcal{J}$ by $\mathcal{B}(K) = \ q^{-1}(K)$.

Prove that these two functions are mutually inverse bijections between sets $\mathcal{J}$ and $\mathcal{K}$.

2. Prove that

[1] Here $J/I \subseteq R/I$ consists of all cosets in R/I with representative in J :

$$J/I \ \stackrel{\text{def}}{=} \ \{j + I;\ j \in J\}.$$

(a) Under the bijection in problem 1.c, the ideal $J = I$ corresponds to the zero ideal $\{0_{R/I}\}$ in R/I. (So, we have I
JJ and we claim that $\mathcal{A}(I) = \{0_{R/I}\}$).

(b) Under the bijection in problem 1.c, if J_1 corresponds to K_1 and J_2 corresponds to K_2, then

$$J_1 \subseteq J_2 \ \ iff \ \ K_1 \subseteq K_2.$$

In other words:

- if $J_1, J_2 \in \mathcal{J}$ and $J_1 \subseteq J_2$ then $\mathcal{A}(J_1) \subseteq \mathcal{A}(J_2)$, i.e., $J_1/I \subseteq J_2/I$;
- if $K_1, K_2 \in \mathcal{K}$ and $K_1 \subseteq K_2$ then $\mathcal{B}(K_1) \subseteq \mathcal{B}(K_2)$, i.e., $q^{-1}(K_1) \subseteq q^{-1}(K_2)$.

(c) Find all ideals J in $\mathbb{Z}$ that contain the ideal $36\mathbb{Z}$. Find all ideals K in $\mathbb{Z}/36\mathbb{Z}$.

3. Let A be a commutative ring, Prove that

(a) $\{0\}$ is a maximal ideal in A iff A is a field.

(b) $\{0\}$ is a prime ideal in A iff A is an integral domain.

4. Use the preceding problems to prove that:

(a) A proper ideal $I \subseteq A$ is maximal iff A/I is a field.

(b) A proper ideal $I \subseteq A$ is prime iff A/I is an integral domain.

5. Use the preceding problem to prove that:

In a commutative ring any maximal ideal is a prime ideal.

6. Let A be a commutative ring. Prove:

(a) If I is a maximal ideal in A and a is an elemet of A which is not in I, then there exist some $i \in I$ and some $x \in A$ such that

$$1 = i + ax.$$

[Hint: From a we construct the principal ideal (a) and then we add it to I to get an ideal $J = I + (a)$. Then $J \subseteq I$ because ... and $J \ni a$ because ... Therefore J is strictly larger than I. Since I isdd maximal, i.e., maximal among proper ideals, we see that the ideal J is not proper. Then, ...]

(b) Use part (a) to prove that:

In a commutative ring any maximal ideal is a prime ideal.

[Hint: I is prime if it is impossible that "$ab \in I$ and $a \notin I$ and $b \notin I$". So, we will suppose that $ab \in I$ and $a \notin I$ and $b \notin I$; and from this assumption we will derive a contradiction. From "$a \notin I$ and $b \notin I$" we find that $1 = i + ax$ and $1 = j + yb$ with $i, j \in I$ and x, y in A. Therefore ...]

7. Let F be a field. We say that a polynomial P is *irreducible* if $\deg(P) > 0$ and P can not be factored into a product of polynomials U, V of lesser degrees

$$P = UV \quad \text{and} \quad \deg(U), \deg(V) < \deg(P).$$

Prove:

(a) If P is irreducible, the ideal (P) in $F[X]$ is maximal.

(b) If P is irreducible, the quotient ring $F[X]/(P)$ is a field.

8. (a) Let F be a field and $P \in F[X]$ be a polynomial of degree 2 or 3. Show that P is irreducible iff it has no roots in F.

(b) Use the above material to prove that

(1) $\mathbb{R}[X]/(X^2 + 1)$ is a field.
(2) $\mathbb{Q}[X]/(X^2 - 2)$ is a field.