

Algebra 411.2

EXAM 3

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All answers should be justified.

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Do any 6 problems, For each problem except problem 6, you can assume that the claims of preceding problems are known (regardless of whether you have solved previous problems).

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The exam is due the last day of the exam period.

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1. Let I be an ideal in a ring R and let $q : R \rightarrow R/I$ be the canonical quotient map.

(a) Any ideal J in R which is larger than I , i.e., $J \supseteq I$, gives rise to an ideal J/I in the quotient ring R/I . Here $J/I \subseteq R/I$ consists of all cosets in R/I with representative in J :

$$J/I \stackrel{\text{def}}{=} \{j + I; j \in J\}.$$

(b) For any ideal K in R/I , its “inverse” in R

$$q^{-1}K \stackrel{\text{def}}{=} \{r \in R; q(r) \in K\} = \{r \in R; r + I \in K\} \subseteq R,$$

is an ideal in R .

(c) The procedures in (a) and (b) are mutually inverse bijections between

- ideals K in R/I and
- ideals J in R that contain I .

2. Prove that

(a) Under the bijection in problem 1.c, the ideal $J = I$ which lies between I and R corresponds to the zero ideal $\{0_{R/I}\}$ in R/I .

(b) Under the bijection in problem 1.c, if J_1 corresponds to K_1 and J_2 corresponds to K_2 , then $J_1 \subseteq J_2$ iff $K_1 \subseteq K_2$.

(c) Find all ideals K in $\mathbb{Z}/36\mathbb{Z}$. Find all ideals J in \mathbb{Z} that contain the ideal $36\mathbb{Z}$.

3. Let A be a commutative ring, Prove that

- (a) $\{0\}$ is a maximal ideal in A iff A is a field.
 (b) $\{0\}$ is a prime ideal in A iff A is an integral domain.

4. Use the preceding problems to prove that:

- (a) A proper ideal $I \subseteq A$ is maximal iff A/I is a field.
 (b) A proper ideal $I \subseteq A$ is prime iff A/I is an integral domain.

5. Use the preceding problem to prove that:

In a commutative ring any maximal ideal is a prime ideal.

6. Let A be a commutative ring. Prove:

(a) If I is a maximal ideal in A and $a \notin I$ then there exist some $i \in I$ and some $x \in A$ such that $1 = i + ax$.

(b) Use part (a) to prove that: *In a commutative ring any maximal ideal is a prime ideal.*

[Hint: If $a \notin I$ and $b \notin I$ then $1 = i + ax$ and $1 = j + yb$ with $i, j \in I$ and x, y in A . Therefore ...]

7. Let F be a field. We say that a polynomial P is *irreducible* if $\deg(P) > 0$ and P can not be factored into a product of polynomials U, V of lesser degrees

$$P = UV \quad \text{and} \quad \deg(U), \deg(V) < \deg(P).$$

Prove:

- (a) If P is irreducible the ideal (P) in $F[X]$ is maximal.
 (b) If P is irreducible the quotient ring $F[X]/(P)$ is a field.

8. (a) Let F be a field and $P \in F[X]$ be a polynomial of degree 2 or 3. Show that P is irreducible iff it has no roots in F .

(b) Use the above material to prove that

- (1) $\mathbb{R}[X]/(X^2 + 1)$ is a field.
 (2) $\mathbb{Q}[X]/(X^2 - 2)$ is a field.