SOME THINGS PHYSICISTS DO

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Contents

1. **Some things physicists do**

To a QFT $\mathcal{T}$ physicists associate a category of *branes* $Br(\mathcal{T})$ and they study $Br(\mathcal{T})$ by associating to each brane $Q \in Br(\mathcal{T})$ a *boundary QFT* $\mathcal{T}_Q$. In simplest cases category $Br(\mathcal{T})$ is something that mathematicians have already considered, such as coherent sheaves on complex manifold or the Fukaya category on a symplectic manifold. In such cases we are interested in results and predictions of physicists resulting from their method of study of a mathematical theory $Br(\mathcal{T})$. Moreover, the theories $\mathcal{T}$ that are mathematically viable are usually simplifications ("*topological twists*") of physical theories which are at the moment beyond our grasp and therefore all the more interesting. For instance, the physical theory of 2-dimensional sigma models has two such topological twists called A-model and B-model and these produce the Fukaya category and the category coherent sheaves.

The complexity of Quantum Field Theories grows with the dimension D of the theory $\mathcal{T}$, for instance $Br(\mathcal{T})$ is really a $(D-1)$-category. So, for $D = 0$ there are no branes. For $D = 1$ $Br(\mathcal{T})$ is a 0-category, i.e., a set, however it is a set with some structures – a Hilbert space. For $D = 2$ it is a category with a structure of a triangulated category. Clearly, for $D > 2$ we have no adequate mathematical formulations ("triangulated 2-categories"?) but there is a rapid progress in this n-categorical direction.

1.0. **A bit of QFT terminology.**

(1) Quantum Field Theories in Lagrangian formalism
(2) Perturbative expansions of Feynman integrals
(3) Dualities
(4) Quantum Field Theories in Hamiltonian formalism
(5) Supersymmetry
(6) Simplifications: *topological twists.*
(7) Example: sigma models.
(8) Branes
(9) Operators
(10) Dimensional reduction via *invariance* and *compactifications.*
(11) Example: gauge theory and S-duality.
(12) Renormalization.

Bellow there a few words on these topics. They are not intended to explain any of these adequately, but rather to establish an unjustified familiarity with some words. The reader may go back to the parts of this material as they become relevant and that "all" will be explained in more details later. Parts (1) and (5-10) will be most relevant.

1.1. **Quantum Field Theories in Lagrangian formalism. A. Quantum Mechanics**. In Classical Mechanics the evolution of a physical system is described by a solution $x(t)$ of a differential equation, Newton's equation of motion $ma = F$, i.e., $\ddot{x} = \frac{1}{m}F(x)$. In Quantum Mechanics the evolution is not precisely predictable, all paths x of evolution of the system are possible however the probability that x will be used is predictable and it grows as x gets closer to the classical solution. In consequence we can not predict the value of *observables* f such as position, velocity, or force acting on the particle at time t, we can only calculate the *expected* value $\langle f \rangle$ of f. The expectation $\langle f \rangle$ is an average of values f would take if it were all possible paths, weighted with the probability that the path in question will be chosen as the evolution of the system.

What is precisely the probability $p(x)$ of using path x? This comes from the Lagrangian formalism in Classical Mechanics – the differential equation of motion is interpreted as criticality equation $dS = 0$ for the *action functional* S on the space of all possible evolution paths (this is the *least action principle*). In Quantum Mechanics $S(x)$ is then used to measure how far x is from the classical solution and the expectation is calculated by the formula

$$\langle f \rangle \;=\; \int_{\text{paths } x} dx\ f(x)\ e^{\frac{i}{\hbar}S(x)}.$$

This involves a new ingredient found experimentally in QM – probability $p(x)$ requires a complex valued refinement $A(x)$ called the amplitude, one has $p(x) = |A(x)|^2$, however amplitudes combine in a simple way while probabilities contain too little information. Another new idea is that the physics depends on the scale $\hbar$ ("*the Planck constant*") on which we observe it. The upshot is that the amplitude depends on the action S and on the Planck constant

$$A(x) = e^{\frac{i}{\hbar}S(x)}.$$

So the above formula measures the expected value of f in the refined sense where probabilities are replaced with amplitudes.

B. Quantum Field Theory. Quantum Field Theory arises as a higher dimensional generalization of classical mechanics. One replaces the time line with a smooth manifold M and generalizes evolution paths (i.e., the maps from the time line to the configuration space of the system) to *fields* on M, i.e., quantities which can be *measured* (i.e., evaluated) at each point of M.

Physics will use this (roughly) in two ways. Manifold M can be the *spacetime* $\mathbb{R}^4$ (or $\mathbb{R}^{10}$...) and then the fields are quantities that can be measured at each point of M such as electric field, magnetic, etc. The other setting is when M is the *worldsheet* of a particle, i.e., the manifold that the particle draws in space time as it involves in time. If a particle is a point (Quantum Mechanics) this is a line. If it is a string (String Theory) the world sheet is a surface, etc.[(1)]

The goal is the same as in QM – to calculate Feynman integral of amplitudes over the space of all fields. The integral has the same form, reflecting the same laws of quantum

[1]The above example of QM had to do only with the second version where M is a a worldsheet. So, the above "justification" of QFT covers only half of its reach. AN example of the other half will be the gauge theory.

interaction. It just gets more difficult as the space of fields gets drastically larger when we increase the dimension of M.

C. Formalism of QFT. The data for a D-dimensional Lagrangian QFT consist of:

(1) a manifold M of dimension D, a choice of a class $\mathcal{F}$ of *fields* on M. These should be objects local in M, say
- functions on M,
- sections of vector bundles on M,
- connections on vector bundles and principal bundles,
- maps from M into a fixed target manifold Σ (this is called sigma model and is the most interesting class of QFT for us). Etc.

(2) A choice of a measure dx on $\mathcal{F}$.
(3) A choice of an action function S on $\mathcal{F}$.

The corresponding QFT is a study of the action function S on the space of fields $\mathcal{F}$, where *study* of S means the study of the *correlation functions* (*correlators*), i.e., the integrals

$$< f > \stackrel{\text{def}}{=} \int_{\mathcal{F}} dx\ f[x]\ e^{iS[x]}, \quad f \in C^{\infty}(\mathcal{F}).$$

The basic correlator is the *partition function*

$$Z \stackrel{\text{def}}{=} < 1_{\mathcal{F}} >= \int_{\mathcal{F}} dx\ e^{iS[x]}.$$

We call these numbers *functions* since the action may depend on some parameters, for instance the Planck scale $\hbar$.

Remarks. (0) *The limit of Quantum Mechanics ate $\hbar \to 0$ is the Classical Mechanics.* As $\hbar$ decreases the oscillations in the integral get faster and the in the limit everything cancels except for the contributions from critical points of S (oscillations die at a critical point). These contributions are called *stationary phase approximation* of oscillating integrals. They come from critical field, i.e., from solutions of the classical equation of motion. Moreover, these contributions are really quantities one finds in Classical mechanics.

(1) *Mathematical content.* From the mathematical point of view the passage from classical physics to QFT corresponds to changing the emphasis from solutions of a certain differential equation $(*)$ in a given space $\mathcal{F}$, to the whole space $\mathcal{F}$ which is now "bent" by the differential equation, i.e., by the corresponding amplitude. The idea here is that both the the equation and the amplitude are described in terms of the *action functional* $S \in C^{\infty}(\mathcal{F})$, interesting equations $(*)$ are criticality equation $dS = 0$ and amplitude is given by as $e^{\frac{i}{\hbar}S(x)}$, $x \in \mathcal{F}$. One can say that we are advancing from the study of critical points of a function S to a study of a function S itself.

The meaning of "bending" the spaces of fields $\mathcal{F}$ by the action S is formalized in the study of expectation values (correlators) of observables (i.e. functions on $\mathcal{F}$).

(2) *Dependence on dimension.* Difficulties increase sharply with the dimension.

(1) $D = 0$ is a part of calculus – the integrals on $\mathbb{R}^n$ or on finite dimensional manifolds.[(2)]
(2) $D = 1$ is QM,
(3) $D = 2$ contains String Theory,
(4) In $D = 4$ one finds the most interesting gauge theory and S-duality which seems to contain the Geometric Langlands as a tiny piece.
(5) In $D = 6$ there seem to be theories which contain the yet unknown Geometric Langlands for surfaces.

Often one thinks that time is one of dimensions of M. Then one breaks the dimension into $D = d + 1$ where $d = D - 1$ is the "space dimension".

D. Making sense of integrals. A priori, this requires choosing a measure dx on $\mathcal{F}$, and checking convergence. Actually, this much can be done only in the simplest cases[(3)] and in general it is million dollar problem.

One can speculate that the correct formulation exists (QM is said to be the most precise science ever with measurements agreeing with theory up to 20 decimal places), but that it is a more sophisticated mathematical object then just a number. Though we do not have a satisfactory theory, what we have is very useful because of many tricks that work for certain classes of examples, say

(1) **Euclidean and Minkowski formulations of Quantum Mechanics.** So far we studied the – physically meaningful – Minkowski formulation. One of the unpleasant features is the difficulty in understanding the oscillating integrals. This can be avoided by switching to integrals

$$< f >_{\hbar}^{Eu} \stackrel{\text{def}}{=} \int_{\mathcal{F}} dx \ f[x] e^{\frac{-1}{\hbar} S[x]}$$

which converge for nice actions such as the free action $S[x] = x^2$. One finds an analytical expression for these integrals and then one evaluates it for imaginary values of the constant $\hbar$.
(2) **Gaussian integrals.** The basic case is when the fields form a Hilbert space $\mathcal{H}$ and the action is $S = \frac{(Ax,x)}{2}$. Then (for a certain normalization of dx)

$$\int_{\mathcal{H}} dx \ e^{-\frac{(Ax,x)}{2}} = \frac{1}{\sqrt{\det(A)}}.$$

More generally, one may deal with fields forming a Riemannian manifold, then convergence hinges on a curvature assumption.

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[2] $D = 0$ was originally a toy example of QFT but it turned out that it does have physical meaning.

[3] The measure for $D = 1$ was constructed by Wiener, since then the first advance was in $D = 2$ by the method of Stochastic Loewner Equation (SLE), see the work of the recent field medalist Werner, and also Kontsevich-.

The Gaussian theories are the *free* theories which describe the undisturbed space before any forces arise. At $\hbar = 0$ we have a classical Newton-Einstein space and for $\hbar > 0$ its "quantum" deformations.

(3) **ζ-regularization.** A trick to interpret some infinite quantities as finite by replacing the apparent value of a zeta function given by a divergent series or product, by the meaningful value of ζ-function obtained by analytic continuation. Say,

$$\sum_{\mathbb{N}} n = \ \zeta(-1) = -\frac{1}{12}.$$

In particular, this is applied to make sense of $det(A)$ when $\dim(\mathcal{H}) = \infty$.

(4) **Perturbative (asymptotic) expansion of integrals** Here, the action S is separated into two parts $S = S_{free} + P$, where S_{free} is a very simple action (Gaussian), and P is viewed as a *small perturbation* of this free action. For this one uses a deformation $S_\varepsilon \stackrel{\text{def}}{=} S\text{free} + \varepsilon P$ of the free action and calculates the corresponding expectations (correlators) $< f >_\varepsilon$ in the sense of calculating the expansion as a series in powers of ε. However,
   (a) This is only an *asymptotic expansion* – usually the radius of convergence is 0.
   (b) The question is then to recover a meaningful function from its asymptotic expansion. The basic problem is that such function is not unique. However, there are some methods of making canonical choices, the basic one is the *Borel summation*. One then has to hope that this choice is really the physically relevant function.

(5) **Semi-classical approximation.** One interprets the integral as a sum of contributions from all critical points of the action S (recall that the critical points are the *classical solutions*). When integrals make sense this procedure gives only the first approximation. However, the method is exact (gives correct answer) in an important class of cases: when there is enough supersymmetry.

(6) **Dimensional Regularization.** In this approach one calculates integrals as certain expressions which depend analytically on the dimension D of the theory. The difficulty in evaluating the original integral directly is reflected in this approach as a pole at D, therefore one then regularizes the analytic expression so that the value at D makes sense.

(7) **Renormalization.** At the moment we will described it only as a systematic science for removing infinities from a computation.

(8) **AKSZ method.** This is an approach towards eliminating the simplest degeneracies of action functionals S (invariance under Lie group – it immediately makes integrals divergent), through embedding the space of fields as a Lagrangian a symplectic supermanifold of *superfields*. (Among other things this is far reaching extension of the more classical BRST-reduction aka symplectic reduction method.)

Remark. (1) should not be confused with the *Wick rotation* which switches the nature of a metric on M – between an ordinary metric, i.e., Euclidean metric, and a Minkowski metric.

1.2. **Perturbative (asymptotic) expansions of Feynman integrals.** Almost all that is known about QFTs comes from the method of expanding theories asymptotically near a classical limit, say for an infinitesimal parameter $\hbar$. The *non-perturbative situations*, i.e., for actual $\hbar$, are more interesting and deeper. One usually starts with the perturbative situation that can be understood by asymptotic expansion, and then tries to add the non-perturbative effects ("*tunneling*"), until the theory seems satisfactory. So, in the study of QFTS the perturbative theories play the role analogous to the local aspects of the analysis on manifolds.

A standard mathematical example of a non-perturbative effect can be found in Morse theory, when the contributions from some fixed points cancel. So, the local contributions to total cohomology have to be glued together using some global information. Here it can be done by counting *instantons* – certain lines between these fixed points So, "instanton corrections" are non-perturbative phenomena.

1.3. **Dualities.** A duality of two QFTs $\mathcal{X}$ and $\mathcal{Y}$ means that in the theories coincide in the sense that are different realizations of the same correlation functions. So, the spaces of fields $\mathcal{F}_\mathcal{X}$ and $\mathcal{F}_\mathcal{Y}$ may be quite different, however there is a correspondence of observables $f_\mathcal{X} \leftrightarrow f_\mathcal{Y}$, such that the expectations are the same: $\langle f_\mathcal{X}\rangle_\mathcal{X} = \langle f_\mathcal{Y}\rangle_\mathcal{Y}$.

An important feature of these dualities is that they may invert the scale parameter $\hbar$ so that the m ore accessible perturbative regime for $\mathcal{X}$ contains information for the non-perturbative regime of $\mathcal{Y}$.

In addition to some dualities which are understood (like T-duality and partly Mirror Symmetry), there are many which are of central interest in QFT and in particular the String theory (like S-duality which seems to contain Langlands duality as a particular degenerate case).

The simplest may be the T-duality that relates sigma models on circles C_r and C_R of radii r and $R = 1/r$. (More generally it relates tori $\prod_1^n C_{r_i}$ and $\prod_1^n C_{1/r_i}$.) It turns out that one can view these circles as dual compact groups U and $\check{U}$, so this is really the abelian Langlands duality.

The dualities manifest on several levels – equality of correlators, equivalence of categories of branes, correspondence of operators.

1.4. **Quantum Field Theories in Hamiltonian formalism.** The basic idea here is that the configuration space for a quantum system is a Hilbert space and this fact is actually deduced from experiments. Roughly, one can view this step as a combination of the standard probabilistic view of Gaussian distributions as aggregate behavior of a stochastic process over many experiments, and a purely new ingredient – the refinement of probability to *amplitude* which is a complex number.

We find that the operations on states and observables are now linear operators. In particular, the evolution of the system is given by a single linear operator that we call the *Hamiltonian operator*.

A. Hamiltonian formalism in Classical Mechanics. In Classical Mechanics the Hamiltonian approach uses the cotangent bundle $T^*\mathcal{C}$ to the configuration space $\mathcal{C}$. [4] It is essential that $T^*\mathcal{C}$ has a canonical geometric structure of a symplectic manifold. The dynamics of the physical system is encoded in the *Hamiltonian function* H on $T^*\mathcal{C}$ – the evolution of the system follows the *Hamiltonian vector field* $\widetilde{H}$ associated to the function H by the symplectic structure.

A justification for the transition to $T^*\mathcal{C}$ is that it provides a large number of symmetries since the Hamiltonian equation is invariant under a large class of transformations of $T^*\mathcal{C}$ – the ones that preserve the Poisson bracket (called canonical transformations or symplectomorphisms). Reducing the problem to the simplest form by a choice of such canonical coordinate change is said to be the strongest method in classical dynamics.

B. Classical Mechanics: Translation from Lagrangian to Hamiltonian formalism. The basic goal of the Classical Mechanics is to determine the position curve $x : \mathbb{R} \to \mathcal{C}$ which describes evolution of the system in the configuration space $\mathcal{C}$. In the Lagrangian approach, Newton's equation of motion is seen as the Euler-Lagrange (i.e., criticality) equation for the velocity curve $v = (x, \dot{x}) : \mathbb{R} \to T\mathcal{C}$ in the tangent bundle to the configuration space. In Hamiltonian approach one instead considers the momentum curve $m : \mathbb{R} \to T^*\mathcal{C}$ in the symplectic variety $T^*\mathcal{C}$ (the "*phase space*").

Here, the momentum curve is the image of the velocity curve under the identification of $T\mathcal{C}$ and $T^*\mathcal{C}$ given by a natural metric on $T\mathcal{C}$ given by a natural metric on M – the kinetic energy. The deeper translation of information between $T\mathcal{C}$ and $T^*\mathcal{C}$ is is given by the *Legendre transform* whose geometric basis is a canonical identification $T(T^*\mathcal{C}) \cong T(T\mathcal{C})$. The Lagrangian encoding of the specific physical setup – the force fields on the configuration space – is given by the action which is the time integral of the *Lagrangian function* L on $T\mathcal{C}$. The Hamiltonian encoding is given by the Hamiltonian function on $T^*\mathcal{C}$ which is the Legendre transform of the Lagrangian function L on $T\mathcal{C}$. Then the Hamiltonian reformulation of Newton's equation says that the evolution of the momentum curve is given by the *Hamiltonian vector field* $\widetilde{H}$ on $T^*\mathcal{C}$ – the vector field associated to the Hamiltonian function H via the canonical symplectic structure on $T^*\mathcal{C}$. In the standard case when L is the difference of the kinetic and potential energy, H is the total energy $H = T + V$ of the system.

[4] It is also called the *canonical formalism* where 'canonical" actually indicates the cotangent bundle setting (as in "canonical transforms").

C. Hamiltonian approach to Quantum Mechanics. From the point of view of physics one may say that passage from Classical Mechanics to Hamiltonian Quantum Mechanics is caused by the experimental fact that the observables can not be adequately described by a bunch of numbers, i.e., by a vector, rather their transformation laws (under changes of coordinates) show that they are Hermitian linear operators on a Hilbert space, the initially observed numbers are just the spectrum of the operator. From the mathematical point of view one passes from Hamiltonian Classical Mechanics to Hamiltonian Quantum Mechanics by the procedure of *deformation quantization*. The basic example is the deformation quantization of the symplectic manifold $T^*\mathcal{C}$ – the commutative algebra $\mathcal{O}(T^*\mathcal{C})$ of functions ("observables") on $T^*\mathcal{C}$, deforms to a non-commutative algebra $D_{\mathcal{C}}(\hbar)$ of $\hbar$-differential operators on $\mathcal{C}$. The mechanism that creates deformation is the symplectic structure on $T^*\mathcal{C}$. Then $D_{\mathcal{C}}(\hbar)$ is viewed as an algebra of operators on the Hilbert space $\mathcal{H}$ which is most naturally chosen as the space $L^2(\mathcal{C})$ of of functions on $\mathcal{C}$. The analysis of $\mathcal{H}$ in this functional interpretation leads to the use of Functional Analysis. The analysis of $\mathcal{H}$ in terms of the physically relevant linear operators is a subject of Representation Theory.

The time evolution of the quantum system is given by the Hamiltonian operator $\widehat{H}$ which is the quantization of the Hamiltonian function on $T^*\mathcal{C}$. A state $v \in \mathcal{H}$ involves in time t to the state $e^{i\widehat{H}}v$.

D. The relation between Lagrangian and Hamiltonian formalisms. The Feynman integrals from the Lagrangian formalisms appear in the Hamiltonian formalisms as matrix coefficients of linear operators. Recall that the Hilbert space of the Hamiltonian approach is in practice realized as a space of functions $L^2(\mathcal{C})$. So, states $u, v \in \mathcal{H}$ are functions and operators such as $e^{i\widehat{H}}$ are then naturally realized as integral operators

$$v(x) \mapsto (e^{i\widehat{H}}v)(y) = \int_{\mathcal{C}} dx \ \mathcal{K}_t(y,x) \ v(x),$$

with some integral kernel $\mathcal{K}_t$. Then the matrix coefficients $\langle u|e^{i\widehat{H}}|v\rangle \stackrel{\text{def}}{=} (e^{i\widehat{H}}v, u)$ are themselves integrals and these are the Feynman integrals that one finds in the Lagrangian formalism.

1.5. **Supersymmetry. A. Hamiltonian picture: bosons and fermions.** Super mathematics entered QM through the separation of particles into bosons ("even") and fermions ("odd") in the Hamiltonian picture of QM. Here the states of a particle p form a Hilbert space $\mathcal{H}$. All possible states of an n-tuple of identical particles $(p_1, ..., p_n)$ then form the Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H}\otimes\cdots\otimes\mathcal{H}$. However, once we forget the names $1, ..., n$ of particles and consider a system of n identical unordered particles, we loose some information and the Hilbert space drops to a subspace $\mathcal{H}^{(n)}\subseteq\mathcal{H}^{\otimes n}$. This is where the "statistics" of two classes of particles differ, If p is a boson then $\mathcal{H}^{(n)}$ is the symmetric power $S^n(\mathcal{H})$ and if p is a fermion then $\mathcal{H}^{(n)}$ is the exterior power $\wedge^n \mathcal{H}$.

To treat both in the same way we say that we attach to a particle a super Hilbert space which for an elementary particle is either purely even (for a boson) or purely odd (for a fermion). Then $\mathcal{H}^{(n)}$ is always $S^n(\mathcal{H})$for the symmetric power taken in the super sense.

B. No go. In the Lagrangian formalism super mathematics appeared as a way around a famous "no go" theorem. This is a result of ... which proves that all reasonable QF theories one can construct have properties which are too restrictive for physically interesting phenomena. Then physicists cheated and allowed fields to be "superfields" which removed the restrictions. The basic case when fields are functions on a manifold now requires manifolds which have even and odd functions, i.e., supermanifolds.

C. Supersymmetries. Introduction of supermanifolds allows one to have supersymmetry ("SUSY"), i.e., odd vector fields on a manifold which are symmetries of some QFT. So we need the space of fields $\mathcal{F}$ to be a supermanifold and the a supersymmetry is an odd vector field on $\mathcal{F}$ which fixes the data of the theory – the action and the Feynman measure. The novel feature of these odd vector fields is that by definition they mix even and odd functions.

For mathematicians supermathematics comes from the "super" rule of what should be the natural identification of tensor products $a\otimes b$ and $b\otimes a$. For physicists supersymmetry is a certain feature of calculations with spinors.

1.6. **Simplifications: topological twists.** The integrability condition for a vector field σ is $[\sigma,\sigma]=0$. This is vacuous in the standard case of even vector fields however for odd objects $[u,v]=uv+vu$, so the condition becomes $\sigma^2=0$, i.e., σ is a differential on the space $\mathcal{O}$ of observables. So a supersymmetry σ will always be assumed to be integrable, and then one can pass to a simpler theory where the observables are given by a smaller space of σ-cohomology of the original observables.

Technically, "topological" refers to the fact that realistic physical theories usually depend on a metric, however this dependence is sometimes eliminated in the passage to $H^*(\sigma,\mathcal{O})$, so the new theory is a Topological QFT, i.e., independent of the metric (actually the simplified theory usually does use the smooth structure of the manifold, so it is not literally situated in topology).

1.7. **Example: sigma models.** In sigma models the fields are maps from the manifold $\Sigma=M$ on which we consider QFT to a fixed target manifold $\mathcal{M}$. Often the idea is that we are interested in some geometry on the target $\mathcal{M}$ and that we use the maps $\Sigma\to\mathcal{M}$ to "probe" this geometry by some low dimensional submanifolds. In mathematics one can think of geodesics as the central ("critical") part of probing a Riemannian manifold with maps from intervals, or of nonstandard analysis which replaces $\mathcal{M}=\mathbb{R}$ with the *hyperreal numbers* $\widehat{\mathbb{R}} \overset{\text{def}}{=} \ = Map(\mathbb{S},\mathbb{R})$ which are essentially sequences in $\mathbb{R}$ that one thinks of as (possibly infinitesimal) processes in $\mathbb{R}$.[5]

[5]Here $\mathbb{S}$ is ...

A. If manifolds come with metrics (Σ, g) and $(\mathcal{M}, G)$ we can consider the volume action or the kinetic action

$$S_{vol}(x) = \ Vol(x(\Sigma)), \qquad S_{kin}(x) = \int_\Sigma \frac{1}{2}||\dot{x}||^2.$$

The critical points in the first case are geodesics and more generally minimal submanifolds, and in the second case the harmonic maps.

B. Poisson model. Sometimes, the fields in a sigma model can be "enhanced maps". For instance a Poisson structure on the target $\mathcal{M}$ gives a *Poisson model* which is a 2-dimensional sigma model. So, $\Sigma = M$ is a surface, the fields are pairs (x, η) of a map $x : \Sigma \to \mathcal{M}$ and $\eta \in \Omega^1(\Sigma, x^*\mathcal{T}^*_{\mathcal{M}})$. The action is

$$S_{Poi}(x, \eta) \stackrel{\text{def}}{=} \int_\Sigma \langle dx, \eta\rangle + \frac{1}{2}\langle \eta\wedge\eta, \pi\rangle.$$

Clearly, we needed the enhancement η in order to probe the Poisson structure π. from the manifold $\Sigma = M$ to a manifold $\mathcal{M}$. The Poisson sigma model is the background of Kontsevich's proof of the deformation quantization conjecture.

In the AKSZ approach to the Poisson model (Cattaneo-Felder), enhanced maps from Σ to $\mathcal{M}$ are interpreted as actual maps between enhanced manifolds – supermanifolds $\Pi T\Sigma$ and $\Pi T^*\mathcal{M}$ (Π means the change of parity in the vector direction), or more precisely between the dg-manifolds $T[1]\Sigma$ and $\mathbb{T}^*[1]\mathcal{M}$ ($[n]$ means the shift of grading by n in the vector direction).

C. A-models and B-models. These are two more complicated theories which arise from a two dimensional sigma model with target a Kähler manifold, by applying topological twist with respect to two supersymmetries that are incarnations of two parts of the Kähler structure – symplectic and complex structures.

1.8. **Branes.** A quantum field theory $\mathcal{T} = (M, \mathcal{F}, S, dx)$ is first formulated on a closed manifold M of dimension D. When we allow M to have boundary, the relevant integrals are not any more over all fields but only over the subset $\mathcal{F}_Q \subseteq \mathcal{F}$ of fields that satisfy some reasonable boundary condition Q. Moreover, we are interested in observing how these fields behave on the boundary and this adds another boundary term to the action. This leads to the notion of branes as "boundary conditions". More precisely, the notion of a brane $\mathcal{B}$ in a theory $\mathcal{T}$ on a manifold M, has a two step meaning:

(1) First, a brane $\mathcal{B}$ contains a condition on how a field ϕ behaves on the boundary ∂M of the source, This has the effect of restricting the space of fields $\mathcal{F}$ in the path integral to the subspace $\mathcal{F}_\mathcal{B}$ of fields that satisfy the condition.
(2) Second, a brane $\mathcal{B}$ contains additional information which defines a *boundary theory*, i.e., an additional term in the action which makes sense for restrictions of fields $\phi \in \mathcal{F}_\mathcal{B}$ to the boundary ∂M.

This gives a QF theory $\mathcal{T}_{\mathcal{B}}$ on M with the space of fields $\mathcal{F}_{\mathcal{B}}\subseteq\mathcal{F}$ and a new action

$$I_{\mathcal{T}_{\mathcal{B}}}(\phi) = I_{\mathcal{T}}(\phi) + I_{\mathcal{B}}(\phi|_{\partial M}).$$

The original action $I_{\mathcal{T}}$ is now called the *bulk action*.

A. Elementary branes and categories of branes. In applications to algebraic or symplectic geometry, it is essential that the branes in a theory $\mathcal{T}$ on M form a *category* $Br(\mathcal{T}, M)$. Actually, this is a $\dim(\partial M)$)-category with some additional structures of an A_∞ category or more traditionally a triangulated category. I will call the branes which come from classical PDE/geometry considerations the *elementary* branes, these will generate the category of branes under the derived category algebra, i.e., by forming complexes (*bound states*) of elementary branes.

B. Examples.

(1) *Dirichlet and Neumann conditions.* In examples close to classical PDEs these conditions are combinations of Von Neumann and Dirichlet boundary conditions. For real valued field ϕ
   - a Dirichlet type condition fixes the value on the boundary $\phi|_{\partial M} = f$ for a fixed field f on ∂M,
   - a Neumann condition eliminates flow across boundary: $\nabla\phi \perp n$ for the normal vector n to ∂M.

   For instance in gauge theory the basic boundary conditions are $F|\partial M = 0$ (Dirichlet condition?), and $\star F|\partial M = 0$ (Neumann condition ?). These are elliptic PD equations.

(2) *2d sigma models.* Here fields are maps $\phi : \Sigma \to \mathcal{M}$ from a surface Σ to a manifold $\mathcal{M}$. Then an elementary brane is a triple $\mathcal{B} = (Y, F, \nabla)$ of
   - a submanifold $Y\subseteq$ *target*,
   - a vector bundle F over Y called the *Chan-Paton bundle* of the brane $\mathcal{B}$, and
   - a connection ∇ on F.

   If we allow F to be a U-bundle we get a sigma model coupled to a gauge theory.

   The component Y gives the condition on fields: ∂M is mapped to Y, so fields are now maps of pairs $(M, \partial M) \to (\mathcal{M}, Y)$. The ϕ-pull-back of the components (F, ∇) to ∂M gives a way of measuring the restriction ϕ on the boundary. For instance, (V, c) gives a gauge theory on $\partial\Sigma$ and one can apply to ϕ the Wilson observables along γ :

(3) *Supersymmetric branes.* In a theory with a supersymmetry Q we are interested in "branes that preserve supersymmetry", i.e., which are compatible with supersymmetry Q (fixed by Q).

   For instance, in a 2d sigma model the supersymmetric branes are given by Gualtieri's notion of *Dirac submanifolds.* For a B-model (a topological twist of a 2d sigma model by a supersymmetry given by a complex structure on the target), a brane (Y, F, ∇)

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is supersymmetric if all three components are holomorphic. The category of supersymmetric branes is believed to be a version of $D^b[\mathcal{C}oh(target)]$. In an A-model (a topological twist of by a supersymmetry given by a sxymplectic structure on the target), submanifold Y has to be coisotropic. The category of supersymmetric branes is believed to be an improvement of the Fukaya category on the target.

(4) For 2d sigma models, the morphisms between branes $\mathcal{B}', \mathcal{B}''$, are called $(\mathcal{B}', \mathcal{B}'')$-*strings*. They are really given by strings stretching in the target stretching between submanifolds Y', Y''.[6]

Remarks. (0) When one uses AKSZ method to reformulate theories in the language of symplectic differential graded manifolds then branes seem to be conditions of a symplectic nature, something like a "restriction to a coisotropic subspace".

(1) It examples above it does not seem quite necessary to consider a topological sigma model in order to define the category of branes. However, the QFT is clearly useful for studying the category of branes. The upshot seems to be that the derived categories of sheaves should be studied by probing with real two-dimensional submanifolds. It seems quite conceivable that one could formalize this probing in some other language besides the Feynman integrals.

1.9. **Operators.** Operators are related to branes and have some similarities. The key features:

- Operators affect QFT.
- Operators have a support which is a submanifold D of the manifold M on which QFT theory lives (as branes do). However, D need not lie on the boundary of M and the operators often have topological invariance – D can be moved without affecting the operator.
- Operators act on branes. (When the support D does not meet ∂M.)

Depending on the size of D we say that we have a point operator, line operator or surface operator,...

There seem to be two main ways to create an operator. In terminology from Statistical Mechanics these are:

- *Order operators* are quantizations of classical expressions $\mathcal{E}$ calculated (observed) on D, by inserting an exponential factor $e^{\mathcal{E}}$ into the path integral. Say, Wilson line operators which carry electric charge are quantization of holonomy.

[6] To have a category we need to compose such strings but then end of a fisrt one need not be the beginning of the second one. Clasically, this would require some latitude in moving strings, someting like homotopy. On the quantum level morphisms are q-strings, i.e., vectors in the Hilbert space obtained by quantizing the space of strings from Y' to Y''. Presumably this makes the ends of strings defined only as probability distributions on Y', Y'' and this allows compositions.

- *Disorder operators* introduce a singularity (disorder) into fields, then the new path integral is over fields with this prescribed singularity along the support D. Say, t'Hooft line operators which carry magnetic charge introduce a singularity along a line,

While the first kind appears naturally in the process of quantizing a classical theory, the second is more surprising since it has no classical existence. Historically, the disorder operators appeared on the quantum level as Fourier transforms of order operators. The first example is electromagnetism, i.e., abelian gauge theory, where t'Hooft operators appeared as Fourier transforms of Wilson operators. This Fourier transform is the abelian case of S-duality, so the extension of the definition of t'Hooft operators to the nonabelian gauge theory can be justified by S-duality conjectures.

There are also operators of combined order-disorder nature, such as Kapustin's Wilson - t'Hooft operators which are *dyons*, i.e., they carry both electric and magnetic charge.

A. Wilson operators in gauge theory. The basic observable in a gauge theory is *holonomy*. A field is a U-bundle E with a connection ∇. The holonomy associated to any path γ on M is the transport operator $Hol_\gamma : E_a \to E_b$ between fibers at endpoints of γ. If γ is a loop then $Hol_\gamma : E_a \to E_a$ is the action of some $\boldsymbol{g} \in {}^E U_a$. In order to interpret it in U (and also to eliminate the choice of a point a in the oriented curve γ), we keep only the conjugacy class hol_γ in U or its image in $U//U = T//W$. Therefore loop γ defines numerical observables $O(hol_\gamma)(E, \nabla)$, given by invariant functions O on U. A standard choice of such O is $Tr{\circ}R$ for some representation R of G.

The observables $O(hol_\gamma)$ are interpreted as *operators* by insertion into path integrals over $\mathcal{A}/\mathcal{G}$, i.e., connections modulo gauge transformations:[7]

$$\int_{\mathcal{A}/\mathcal{G}} dA\ e^{-I(A)}{\cdot}e^{O(hol_\gamma(A))}.$$

B. 't Hooftzzz operators. Transversally to a line L in a 4d manifold M we have $\mathbb{R}^3$. A *singularity* of fields Φ along L will be described by a particular field Φ_0 on $\mathbb{R}^3 - 0$, then we ask that Φ "behaves as a multiple of Φ_0 near L". The relevant singularities are *conical*, i.e., Φ_0 is *scale invariant.* So, Φ_0 lives on $(R^3 - 0)/\mathbb{R}_+^* = S^2$.

Moreover, the singularities of interest are fields that are solutions of YM equations. So, they are given by solutions of YM on $S^2 = \mathbb{CP}^1$, i.e., G-bundles on $\mathbb{P}^1$. However,

Lemma. Isomorphism classes of G-bundles on $\mathbb{P}^1$ are in a natural bijection with $Irr(\check{G})$.

Proof. The LHS can be interpreted as $G(\mathbb{C}[z^{-1}])\backslash G(\mathbb{C}((z)))/G(\mathbb{C}[z]])$ (transition functions near 0 modulo the ones that extend across 0 or across $\mathbb{P}^1 - \{0\}$). So, these are the orbits of $G(\mathbb{C}[z^{-1}])$ on the loop Grassmannian $\mathcal{G}$. There is a dual stratification of $\mathcal{G}$ (in the sense

[7] The partition function of the gauge theory contains only $I(A)$ so it depends only on curvature, while in the new theory integrand depends more subtly on the connection itself.

of Morse theory, i.e., stable and unstable manifolds), given by orbits of $G(\mathbb{C}[[z]])$, but these are indexed by $Irr(\check{G})$.

C. Action of operators on branes. For instance, in 2d we can consider a surface M with a boundary component σ and a line L that runs parallel to σ. A line operator $\mathcal{O}$ on L combines with a brane $\mathcal{B}$ on σ into a new brane $\mathcal{B}' = \mathcal{O}\mathcal{B}$.(8) Effectively, one finds that the combined influence of $\mathfrak{X}$ and $\mathcal{B}$ on path integrals, can be considered as the influence of a new brane $\mathcal{B}'$.(9)

Notice that as branes form a category, operators act as functors between categories of branes.

Remark. Of course if want an action on branes of a certain kind – with a certain supersymmetry , then we need our topological operator to have the same kind of supersymmetry.

D. Surface operators in 4d gauge theory. When one extends the unramified GL to the tamely ramified case the moduli of G-bundles on C is replaced by the moduli of *parabolic G-bundles* and similarly one needs to insert *parabolic* when talking of Higgs fields and the *ramified* when talking of local systems. It is known that these are again faces of certain hyperkähler manifold, the *parabolic* Hitchin moduli [Simpson]. The question arises whether the 2d sigma model with target the parabolic Hitchin moduli also appears as a 2d limit of some gauge theory in 4d. Gukov-Witten found that one can use the same 4d gauge theory in 4d but at a ramification point one needs to insert a surface operator.

So, in 4d we consider a manifold $M = \Sigma\times C$ which is a product of Riemann surfaces, one of which is our curve C. A ramification point $p \in C$ then defines a surface $D = \Sigma\times p\subseteq M$ which is the support of a surface operator $\mathcal{S}$. This is a disorder operator which introduces a singularity along D described by a quadruple $(\alpha, \beta, \gamma, \eta) \in T\times\mathfrak{t}\times\mathfrak{t}\times\check{T}$. Here, $(\alpha, \beta, \gamma 0$ are geometric parameters which appear in the work of Simpson. However, η is a "quantum" parameter (complexified Kaḧler parameter or B-field). It does not appear in classical considerations of the algebraic geometry of relevant spaces such as moduli of parabolic bundles, however, by now, the role of B-fields is more familiar in considerations of derived categories of sheaves on geometric objects.

These surface operators do not act on branes. The reason is that $\partial D = \partial\Sigma\times p$ lies in $\partial M = \partial\Sigma\times C$. in

1.10. **Dimensional reduction via invariance and compactifications.** A dimensional reduction of a theory $\mathcal{T}$ on a manifold M appears when M is a product $\Sigma\times C$ (or maybe a M is a bundle over Σ with fiber C).

8One can describe this either as a limit of moving L towards σ, or as the macroscopic effect of moves far away until L and σ blend.

9This is easy to see for Wilson operators.

A. Dimensional reduction via invariance This is the simplest dimensional reduction. Inside the fields $\mathcal{F}$ of the theory $\mathcal{T}$ one looks at the subspace $\mathcal{F}'$ of fields which are ion some sense constant along C, say if C is a group then we look at C-equivariant fields. Then $\mathcal{F}'$ can be though of as fields on Σ, however they have are likely to have some "complicated" structure reflecting the fact that they naturally appear on a higher dimensional manifold. The action I on $\mathcal{F}$ restricts to $\mathcal{F}'$ and therefore defines a new theory $\mathcal{T}'$ on Σ (actually one also needs a measure on $\mathcal{F}'$).

B. Dimensional reduction via compactifications Again, $\mathcal{T}$ is a QFT on a product of Riemannian manifolds $M = \Sigma \times C$. This time we rescale the metric on C so that

$$\Sigma \textit{ is much larger then } C.$$

For this to make sense we ask that C is compact so that its total size may be reduced by this rescaling.[10] Intuitively in the limit C becomes a point (with some structure) and theory $\mathcal{T}$ becomes a theory $\mathcal{T}_C$ on Σ which contains some information about C.[11]

The limiting process makes action *large* for fields that change much along C. According to the stationary phase principle. the path integral localizes to fields which minimize the action in the C-direction and change freely in the Σ-direction. These are the fields of the new theory $\mathcal{T}_C$ on Σ.

1.11. **Example: gauge theory and S-duality.** Gauge theory is also called Young-Mills theory. The *data* for a gauge theory are: (i) a Riemannian manifold M with a metric g, and (ii) a compact group U.
fttt The standard notation for physicists is G instead of U. For us G is usually a complex algebraic group arising as the complexification of U.

The fields are U-connections on M, i.e., pairs (E, ∇) of an U-torsor E over M and a U-connection ∇ on E. Physicists calculate locally so they locally choose a trivialization of E, then $\nabla = d + A$ for the trivial connection d and a 1-form $A \in \Omega^1(M, \mathfrak{u})$ with values in the Lie algebra $\mathfrak{u}$ of U. From this point of view they say that the fields are 1-forms A.[12]

[10] Finite volume would also work?

[11] In physics this is called Kaluza-Klein mechanism. Famously, it is supposed to explain why we perceive only 4 dimensions though the spacetime should be 10 dimensional – the other 6 dimensions are wrapped up on a compact manifold C which is too small to observe directly. We observe C only indirectly, by noticing that the physics of our world can be explained in simple terms if we postulate that this it is a combination of a simpler theory $\mathcal{T}$ in 10 dimensions and geometric features of a 3 dimensional Calabi-Yau manifold C.

[12] The first step in a physics computation are local calculations. After that, in the second step they account for the nonlocal effects. For instance calculations with a "field" A are actually valid local calculations, but one still needs to account for global the effects of changing the frame (trivialization) of E. Physicists say that a gauge field A gives a *covariant derivative* $D = d + A$, actually D is the real connection ∇ in question, while A and d is just its local components defined for each local trivialization of P.

A. Young-Mills action. There is a "kinetic" action where the role of velocity is replaced by the curvature F

$$S_{kin}(E,\phi) = \int_M \frac{1}{2}||F||^2 = \int_M \frac{1}{2}Tr(F\wedge\star F).$$

Here $\star$ is the Hodge star operator defined by the metric g on M and Tr denotes a *negative definite metric* on U.[(13)] U to be compact. Actually, the metric will be uniquely normalized so that ch_2 takes all integer values on $M = S^4$.

If $\dim(M) = 4$ there is also a *topological action*

$$S_{top}(E,\phi) = \int_M \frac{1}{2}Tr(F\wedge F)$$

which really depends only on the torsor E (not on the metric g and not on the connection A) and is an integer for closed manifolds M.[(14)]

The standard situation in QFT is that one studies a *family* of interesting actions, obtained as linear combinations of elementary actions. The coefficients are called the *coupling constants* as they describe how various ingredients are coupled together. The standard way of writing these constants in gauge-theory is

$$S = \frac{1}{4e^2}S_{kin} + i\frac{\theta}{2\pi}S_{top}.$$

Here e is related to the electron and θ is called the θ-angle.

B. S-duality. The complexified gauge coupling is then defined as

$$\tau \stackrel{\text{def}}{=} \frac{\theta}{2\pi} + i\frac{1}{4e^2}.$$

Notice that it lies in the upper half plane $\mathbb{H}\subseteq\mathbb{C}$.

The *S-duality conjecture* (*Montonen-Olive conjecture*) is an equivalence of gauge theories for (U,τ) and $(\check{U}, -\frac{1}{l_\mathfrak{u}\cdot\tau})$ where $l_\mathfrak{u}$ is the lacing of a simple Lie algebra $\mathfrak{u}$, i.e., the square of the ratio of lengths of longest and shortest roots. (For $(U = U(n)$ the lacing is 1 so we can forget it.)

This is a part of a larger group of symmetries which acts on the complexified couplings, i.e., on the upper half plane $\mathbb{H}$. This *S-duality group* is either $SL_2(\mathbb{Z})$, or its congruence

In general, physicists seem to study a situation through a series of considerations of increasingly more subtle ingredients. To a mathematician it may seem that they do not stop to put these observations together and account for all interactions of these ingredients. However, this may be an incorrect impression since there products usually turn out quite reliable.

[13] The existence of such metric forces the gauge group

[14] This is really the second Chern class of the torsor E which can be calculated in terms of any connection A on E by

$$ch_2(E) \stackrel{\text{def}}{=} \frac{-1}{8\pi^2}\int_M TrF\wedge F.$$

subgroup of (when the lacing is not 1). Here "S" refers to a generator $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of $SL_2(\mathbb{Z})$. The other generator $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ acts simply by a 2π shift in the theta angle θ.

Remark. As $SL_2(\mathbb{Z})\backslash\mathbb{H}$ is the moduli of complex elliptic curves, the conjecture roughly says that the gauge theory does not depend on the coupling τ so much as on the associated elliptic curve E_τ. Actually, T-invariance means that the theory depends on $q \overset{\text{def}}{=} e^{2\pi i\tau}$ and $E_\tau = \mathbb{C}^*/q^{\mathbb{Z}}$, however

Question. Does this mean that the theory should depend on the elliptic group G_τ (an unknown analogue of a quantum group). Then G_τ should be $\check{G}_{-1/\tau}$?

One of the formulations of S-duality is an equivalence of categories of branes

$$Br(M, U, \tau) \;\cong\; Br(M, \check{U}, -\frac{1}{l_{\mathfrak{u}}{\cdot}\tau}).$$

Notice that since M is four dimensional, from the point of view of mathematicians the setting for a complete treatment of this situation would be a notion of a "triangulated 3-category".

The origin of S-duality is the case $U = U(1) = \check{U}$, this is the Electro-Magnetic Duality, It is first seen on the classical level as a symmetry of Maxwell's differential equations. The quantization requires a formulation in terms $U(1)$-torsors and the symmetry of equations This case is well understood – S-duality in the abelian case is a Fourier transform.

Physicists view the S-duality conjecture as deep and central. It is studied through massive numerical experiments. Kapustin-Witten noticed that certain two step simplification of S-duality (a topological twist plus a compactification to two dimensions), is related to the geometric Langlands conjecture over complex numbers. In their approach S-duality is the deeper half of a conjectural construction of Langlands correspondence.

C. Super gauge theories. The above YM theory is found not to be quite satisfactory for physics and this lead to introduction of super gauge theories. For instance in 4d, the gauge theory is conformal classically but not quantum mechanically. There are many ways to correct this by adding additional fields and some of the added fields are usually odd. For instance, while the fields of a standard gauge theory are connections A ("*gauge field*"), a typical field in a super gauge theory is a pair (A, λ) where fermion λ ("superpartner"),[15] is an even spinor with "values in $\mathfrak{g}$", i.e.,

$$\lambda \in \Gamma(M,\ {}^{E}\mathfrak{g}\otimes\mathcal{S}^+).$$

Besides the *data* for a gauge theory: are (i) a manifold M with a metric g, and (ii) a compact group U; for a super gauge theory one also needs (iii) a spin bundle $\mathcal{S}$ on M.

[15] The attitude is often that superfields are of less importance then the original fields, a kind of fill-in that makes the original even fields behave correctly.

D. Maximal dimensions for Supersymmetric or Superconformal gauge theories. By a theorem of Nahm, dimension 10 is a maximal dimension for a *supersymmetric* gauge theory. Moreover, such theory in dimension 10 is essentially unique. It's special role can be argued to be related to octonions. It is used to produce supersymmetric gauge theories in smaller dimensions by various dimensional reductions.

Let us go a step further and require that the theory be invariant under (super)conformal transformations of the (super)spacetime, Nahm also observed that nontrivial *superconformal* gauge theories theories exist at most for the dimensions $D \leq 6$.[(16)]

Until recently, it was expected that nontrivial theories above 4d in fact do not exist. However, there seems to exist superconformal theories in $D = 6$, constructed as quantizations of ADE singularities. In this setting Witten sees key features of the GL program extended to complex dimension 2 (representations of loop groups as Hecke-like operators). In order to resolve the absence of the topological term outside 4d, the theories of interest should be *self-dual* in the sense that they use only the connections with a self dual curvature. The existence of a self-dual theory for $D = 6$ would imply the S-duality in 4d. (This is one of the ways S-duality can be derived from String Theory.)

1.12. **Renormalization.** There is a number of renormalization prescriptions that strive to systematically cancel the infinities that appear in computations. One of these, the *Minimal Subtraction scheme* with *Dimensional Regularization* has recently been understood mathematically by Kreimer, Connes and Marcolli. It turns out to be related to the Birkhoff factorizations in loop groups, to the Riemann-Hilbert correspondence and conjecturally to motives.

Appendix A. **Electro-magnetic Duality and S-duality**

This is the origin and the baby case of S-duality. On the classical level electromagnetism is decribed by Maxwell's equations. On the quantum level, electromagnetism will be described in terms of the $U(1)$ gauge theory. The S-duality is then the extension of the EM duality to the gauge theory for the general group G.

A.1. **Classical theory of electromagnetism: fields and Maxwell's equations.**

A.1.1. *Electro-magnetic fields.*

- Electric field E is a vectorfield on $\mathbb{R}^3$ or $\mathbb{R}^{3,1}$.
- Magnetic field B is a 1-form on $\mathbb{R}^3$ or $\mathbb{R}^{3,1}$.
- Relativistically, E and B combine into a 2-form on $\mathbb{R}^3$

$$F \stackrel{\text{def}}{=} dt \wedge E \ + \ \star B,$$

[16] In fact Nahm notices that the existence of superconformal theories requires some special isomorphisms of groups which are come from coincidences in the ABCDEFG families of simple groups.

for the Hodge star operator $\star$ on $\mathbb{R}^3$.

A.1.2. *Maxwell's equations in vacuum.* The equations are

$$dF = 0 \quad \text{and} \quad d(\star F) = 0.$$

A.1.3. *Classical duality.* The original 19th century version of Electro-magnetic Duality is the obvious symmetry of Maxwell's equations

$$F \mapsto \star F \quad \text{and} \quad \star F \mapsto -F.$$

The signs are chosen so that the symmetry is $\star$-equivariant (use $\star^2 = -1$).

A.2. **Quantum mechanical setting for electromagnetism.** In the QM formulation the above obvious symmetry between F and $\star F$ is broken:

- Equation $dF = 0$ is interpreted as

  F is the curvature ∇^2 of a connection $\nabla = d + A$ on a line bundle L over the spacetime M_4.

  This interpretations is the solution of $dF = 0$ and this equation now disappears (it becomes trivial: "Bianchi identity").
- In this geometric setting physics consists of one differential equation $d(\star F) = 0$, so in the Lagrangian approach this has to be the criticality equation for a Lagrangian.

A.2.1. *Kinetic action.* In order to interpret Maxwell 's second equation $d(\star F) = 0$ as the criticality equation of an action, we integrate it to a "kinetic" action

$$I_{kin} \stackrel{\text{def}}{=} \int_{M_4} ||F||^2 = \int_{M_4} F \wedge \star F.$$

This is analogous to harmonic maps, the difference is only that one uses the "kinetic energy" of a connection rather then a "kinetic energy" of a map.

A.2.2. *Topological action.* However, in 4d there is a possibility to enrich the action with a topological term

$$I_{top} \stackrel{\text{def}}{=} \int_{M_4} F \wedge F = i\theta \int_{M_4} c_1(L)^2.$$

Remark. The topological term does not influence the criticality equations (it is locally constant in F !), hence it does not influence the classical solutions.

A.2.3. *Total action.* The total action is a combination

$$I \stackrel{\text{def}}{=} \frac{1}{4e^2} I_{kin} + i \frac{\theta}{(2\pi)^2} I_{top} = \frac{1}{4e^2} \int_{M_4} F \wedge \star F + i \frac{\theta}{(2\pi)^2} \int_{M_4} F \wedge F.$$

A.2.4. *Couplings* e, θ. Physically, e is the charge of an electron. θ is an angle variable. We combine them into one complex coupling

$$\tau \stackrel{\text{def}}{=} \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}.$$

Remark. Couplings are a purely *quantum phenomena*. Notice that they do not influence the classical solutions, i.e., the criticality equations (remark A.2.1). However, they certainly influence the action and hence the quantum theory.

A.2.5. *Path integral.* The partition function is an integral over the space $\mathcal{A}$ of connections modulo the group $\mathcal{G}$ of gauge transformations

$$Z = \langle 1 \rangle \stackrel{\text{def}}{=} \int_{\mathcal{A}/\mathcal{G}} dA \quad e^{-I(A)}.$$

Notice that the integral depends on connectionns not only on curvatre – the integrand depends only on the curvature F, however the measure does depend on A.

A.3. **Quantum Electro-Magnetic Duality.** The obvious symmetry of the classical picture of EM force, i.e., symmetry of Maxwell's equations, does not disappear in the in the quantum mechanical setting but here it becomes a more complicated symmetry denoted S.

(1) For one thing the new symmetry S is non-clasical since it notices the coupling. It acts on the complex coupling by $S(\tau) = -1/\tau$.
(2) Still, S exchanges F and $\star F$, so can be viewed as the quantum version of the classical EM duality.
(3) S is given by the *Fourier transform* in the space of all connections. The action on τ is roughly a version of the Fourier transform formula

$$\int \frac{dx}{\sqrt{2\pi}} e^{ixy} \, e^{-\lambda x^2/2} = e^{-\lambda^{-1} y^2/2}$$

which inverts λ.
(4) S exchanges F and $\star F$, so it can be viewed as the quantum version of the classical EM duality.
(5) There is an additional classical synmmetry T which shifts the θ-angle $\theta \mapsto \theta + 2\pi$. Equivalently, $T(\tau) = \tau + 1$. This affects the action but not the path integral.[(17)]
(6) Quantum symmetry S combines with a classical synmmetry T to generate a symmetry group $\Gamma \cong SL_2(\mathbb{Z})$ called the S-duality group (or Hecke group).

[17] At least not when M is closed, then the integral produces an integer.

A.4. **S-duality.** A conjectural quantum symmetry S is a generalization of the quantum EM duality. It arises when one considers gauge theory for arbitrary compact groups U. Then the electric charges are irreducible represntations of U and the magnetic charges are irreducible represntations of $\check{U}$.[(18)]

Conjecture. There is a quantum symmetry S which

(1) "inverts" τ by $S(\tau) = -1/l_U\tau$,[(19)]
(2) exchanges U and $\check{U}$ and therefore also the electric and magnetic charges.
(3) combines with a classical synmmetry T (shift of the θ-angle), to give a finite index subgroup Γ of $SL_2(\mathbb{Z})$ – the S-duality group of U.

One can try to make this more precise by introducing two copies $\mathbb{H}$ and $\mathbb{H}'$ of the upper half plane, related to U and $\check{U}$. Then T acts the same on both by $\tau \mapsto \tau + 1$ and $\tau' \mapsto \tau' + 1$, However, S exchanges the two copies by

$$S = \begin{pmatrix} 0 & 1/\sqrt{n} \\ -\sqrt{n} & 0 \end{pmatrix}, \quad \text{i.e.,} \quad S(\tau) = \frac{-1}{n\tau'}, \quad S(\tau') = \frac{-1}{n\tau}.$$

If $\check{\mathfrak{g}} \cong \mathfrak{g}$ one can pretend that $\mathbb{H}' = \mathbb{H}$.

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[18] In the case $U \cong U(1) \cong \check{U}$ both kinds of charges can be thought of simply as integers, i.e., integer multiples of one basic charge.

[19] l_U is the lacing for both $\mathfrak{u}$ and $\check{\mathfrak{u}}$.