D-MODULE LANGLANDS CONJECTURE IN POSITIVE CHARACTERISTIC
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Contents

1. Crystalline Differential operators

Let X be a smooth algebraic variety over a closed field $\Bbbk$ of characteristic p. We consider the sheaf of *crystalline differential operators* $\mathcal{D}_X \overset{\text{def}}{=} U_{\mathcal{O}_X}(\mathcal{T}_X)$.

1.0.1. *Lemma.* (a) $Z(\mathcal{D}_X) = \mathcal{O}_{T^*X^{(1)}}$

(b) $\mathcal{D}_X$ is Azumaya over $T^*X^{(1)}$.

(c) Azumaya algebra $\mathcal{D}_X$ splits over the conormal Lagrangian $T^*_Y X$ for any submanifold $Y \subseteq X$.

1.0.2. *The support and p-curvature of a D-module.* We consider $\mathcal{D}_X$ as an algebra sheaf over $T^*X^{(1)}$. Then $\mathcal{D}_X$-modules localize on $T^*X^{(1)}$.

The *p-curvature* of a D-module $\mathcal{F}$ is defined as the action of the center

$$\psi : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X^{(1)}}, \quad \langle \psi f, \partial^{(1)} \rangle \overset{\text{def}}{=} (\partial^p - \partial^{[p]}) f.$$

One can think of p-curvature as a map

$$\psi_{\mathcal{F}} \ \in \ \mathcal{E}nd_{\mathcal{D}}(\mathcal{F}) \otimes \Omega^1_{X^{(1)}}.$$

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Let us say that *projectively p-flat* D-modules are those with scalar p-curvature

$$\psi_{\mathcal{F}} \in 1_{\mathcal{F}} \otimes \Omega^1_{X^{(1)}}.$$

In this case the p-curvature ψ is a 1-form on $X^{(1)}$ and $\mathrm{supp}(\mathcal{F})$ lies in the graph $\Gamma_\psi \subseteq T^* X^{(1)}$.

Remark. This looks like microlocalization however, it is not the standard one but rather alike the A-brane version of Kapustin-Nadler-Zaslow that is used in the Kapustin-Witten approach to geometric Langlands. The traditional microsupport is conelike (however there is a delocalized version by Hormander ...?).

1.0.3. *Splitting gives a commutative picture.*

1.1. **Critical quantization.**

2. **Gerbs**

2.0.1. *The classifying stack $B(G)$.* Let G be a group bundle over X. When G acts on a scheme Y over X, we get the quotient stack Y/G which is determined (and defined) by decribing $\mathrm{Hom}(S, Y/G)$ for any scheme S. The sapce of maps $\mathrm{Hom}(S, Y/G)$ is the category of pairs (P, F) where P is a G-torsor over S and $F : P \to Y$ is a G-map.

We are interested in the classifying stack $B(G) \overset{\text{def}}{=} X/G$.

Lemma. Let G be a group bundle over X.

(a) There is a canonical map $B(G) \overset{\text{def}}{=} X/G \to X$.

(b) The sheaf of sections $\underline{B(G)}$ of $B(G)$ over X is the sheaf of categories $\mathcal{T}ors_G$ of G-torsors.

(c) The category $\mathcal{C}oh[B(G)]$ of coherent sheaves on the stack $B(G)$ is the category $\mathcal{C}oh_G(X)$ of G-equivaraiant coherent sheaves on X.

Proof. (a) Having a map $X/G \to X$ means having a transformation of functors $\mathrm{Hom}(-, X/A) \to \mathrm{Hom}(-, X)$. For any scheme S, to any map $(S \leftarrow P \xrightarrow{F} X) : \ S \to X/G$ one canonically associates a map $\overline{F} : S \to X$, the factorization of F which exists since the G-action on X is trivial.

(b) The sheaf of sections $\underline{X/G}$ of $X/G \to X$ is given by $\underline{X/G}(U) = \Gamma(U, X/G) = Map_U(U, X/G)$. Now, a map $U \to X/G$ is a pairs of a G-torsor $P \xrightarrow{\alpha} U$ over U and a G-map $F : P \to X$. This is a section iff it is a map over U. The last condition means that F maps P to U and that this map coincides with the structure map α. So,

$$\underline{X/G} = \mathcal{T}ors(G).$$

Corollary. For a commutative group A over X, the stack $B(A)$ is a group stack over X.

Proof. When A is commutative, the sheaf of sections $\underline{B(A)} = \mathcal{T}ors_A$ hasa structure of an "abelian group category" (Picard category) – since left and right A-torsors coincide we have multipication $P{\times}_A Q$ of A-torsors.

2.0.2. *Gerbs.* Let A be a commutative group bundle over X. An A-*gerb* $\mathfrak{X} \to X$ is a torsor for the group stack $B(A)$. So, locally $\mathfrak{X} \cong B(A)$ and $\mathfrak{X}$ is again a stack over X.

2.0.3. *Lemma.* A-gerbs $\mathfrak{X}/X$ are the same as $\mathcal{T}ors_A$-torsors.

Proof. This is just the correspondence of spaces and their sheaves of sections – an A-gerb $\mathfrak{X}$ is a torsor for $B(A)$, so its sheaf of sections $\underline{\mathfrak{X}}$ is a torsor for the sheaf of groups $\underline{X/A} = \mathcal{T}ors(A)$.

2.0.4. *Gerbes and Azumaya algebras.* From now on we only consider the group $A = G_m$ and "gerb" means a G_m-gerb.

Lemma. (a) The category of coherent sheaves on a gerb $\mathfrak{A}$ over X is $\mathbb{Z}$-graded

$$\mathcal{C}oh(\mathfrak{A}) = \ \oplus_{n\in\mathbb{Z}}\ \mathcal{C}oh(\mathfrak{A})_n.$$

(b) Any Azumaya algebra A/X defines a gerb $\mathfrak{A}/X$. Its sheaf of sections is the sheaf of categories $\underline{\mathfrak{A}}(U)$ of splitting bundles of the Azumaya algebra A on U.

Proof. (a) We know that $\mathcal{C}oh(X/G_m) = \mathcal{C}oh_{G_m}(X)$, since G_m acts trivially on X this is just a sum of copies $\mathcal{C}oh(X)_n$ of the category $\mathcal{C}oh(X)$, where G_m acts by z^n on the $n\theta$ power. Since a gerb is locally isomorphic to the classifying stack $B(G_m)$, $\mathcal{C}oh(\mathfrak{X})$ inherits the grading.

(b) $\underline{\mathfrak{A}}(U)$ torsor for the group category $\mathcal{T}ors_{G_m}$ because any two splittings on the same U differ by tensoring with a line bundle.

2.0.5. *Fake abelianization of Azumaya algebras.* For an Azumaya algebra A over X, the category of coherent A-modules has a commutative description where all subtlety is stored into the geometry of the associated stack $\mathfrak{A}$:

Lemma. $\mathcal{C}oh(A) \cong \mathcal{C}oh(\mathfrak{A})_1$.

2.0.6. *Grouplike gerbs and Azumaya algebras.* Let us say that for an A-gerb $\mathfrak{G}$ over a group G, a *compatibility structure* for the group structure $G{\times}G \xrightarrow{m} G \xleftarrow{i} 1_G$ is a pair of an isomorphism of A-gerbs on $G{\times}G$

$$\iota:\ m^*\mathfrak{G}\ \cong \mathfrak{G}\boxtimes_A\mathfrak{G}$$

and a trivialization $i : \mathfrak{G}|_{1_G}\xrightarrow{\cong} B(A)$, satisfying certain consistency properties.

Lemma. For an A-gerb $\mathfrak{G}$ over a group G, a *compatibility structure* makes $\mathfrak{G}$ into a group stack. It is an extension of group stacks

$$0 \to B(A) \to \mathfrak{G} \to G \to 0.$$

Proof. The inclusion $B(A) \to \mathfrak{G}$ is the trivialization $i : \mathfrak{G}|_{1_G} \xrightarrow{\cong} B(A)$. The map $\mathfrak{G} \to G$ is the quotient map $\mathfrak{G} \to \mathfrak{G}/B(A) \cong G$ for the $B(A)$-torsor $\mathfrak{G}$ over G.

3. Geometric Langlands conjecture

4. Geometric Langlands conjecture for D-modules in positive characteristic