GLOBAL UNRAMIFIED GEOMETRIC LANGLANDS CONJECTURE WITTEN-KAPUSTIN

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Contents

1. Global Unramified Geometric Langlands conjecture

1.1. **Geometric Langlands conjectures.** These are algebro-geometric expectations with roots in Number Theory. Here *global* means that conjectures concern a projective curve over a closed field while *unramified* means that we consider local systems which are defined on the whole curve, i.e., have no singularities. The conjectures have two main forms – (a) categorical equivalence, and (b) existence of Hecke eigensheaves.

1.1.1. *The data.* Let C be a smooth projective curve over a closed field $\Bbbk$. Choose a reductive algebraic $\Bbbk$-group G, it turns out that such groups come in *Langlands dual pairs* $G, \check{G}$. We will attach to each of these groups one moduli of objects on C and a category of sheaves on this moduli. We will be interested in the case $\Bbbk = \mathbb{C}$ which allows the differential geometric treatment of Kapustin and Witten. For the general $\Bbbk$ the story is technically different as it uses the language of l-adic cohomology.

First, let $\mathrm{Bun}_G(C)$ be the moduli stack of G-torsors (principal G-bundles) over C. If $G = GL_n$ then G-torsors are equivalent to rank n vector bundles, so $\mathrm{Bun}_G(C)$ is the same as the moduli $\mathrm{Vec}_n(C)$ of such vector bundles on C.

Next, the moduli stack $\mathcal{LS}_G(C)$ of G-local systems on C classifies (if the characteristic of $\Bbbk$ is zero), pairs of a G-torsor P and a connection ∇ on P (by that we mean a G-invariant connection).[(1)] If $\Bbbk = \mathbb{C}$ we can think of G-local systems as group maps $\pi_1(C) \to G$. We will attach this moduli to $\check{G}$, so we get $\mathcal{LS}_{\check{G}}(C)$.

1.1.2. *Conjectures.* The Geometric Langlands conjecture of Beilinson and Drinfeld says that $\check{G}$-local systems on C can be encoded as D-modules on the stack moduli $\mathrm{Bun}_G(C)$. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators on X.

In the deeper categorical form, the moduli $\mathcal{LS}_{\check{G}}(C)$ and $\mathrm{Bun}_G(C)$ are related by a kind of a Fourier transform exchanging coherent sheaves on $\mathcal{LS}_{\check{G}}(C)$ and D-modules on $BunG$. The fact that the duality also exchanges G and its *Langlands dual* $\check{G}$ is alike the fact that the Fourier transform exchanges functions on dual vector spaces.

1.1.3. *Conjecture.* (a) There is a canonical equivalence of triangulated categories

$$\mathcal{L} \;:\; D^b[\mathcal{C}oh(\mathcal{LS}_{\check{G}}(C))] \;\cong\; D^b[mod(\mathcal{D}_{\mathrm{Bun}_G(C)})].$$

We call $\mathcal{L}$ the *Langlands transform.*

(b) For any $\check{G}$-local system $\mathcal{E}$ on the curve C, there is a unique Hecke eigen-D-module $\check{\mathcal{E}}$ on $\mathrm{Bun}_G(C)$, with eigenvalue $\mathcal{E}$.

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[1] For arbitrary $\Bbbk$ there is still a notion of a G-local system over C, these are the l-adic local systems in etale topology. In this case we use the $\overline{\mathbb{Q}_l}$-form $G_{\overline{\mathbb{Q}_l}}$ of G, so for $G = GL_n$ a G-local system means a ordinary rank n l-adic local system. For a general G a G-local system $\mathcal{L}$ is then viewed in Tannakian terms as a family of ordinary l-adic local systems $\mathcal{L}_V$ indexed by representations V of $G_{\overline{\mathbb{Q}_l}}$, i.e., $\mathcal{L}_V$ plays the role of associated bundle for a $G_{\overline{\mathbb{Q}_l}}$-torsor $\mathcal{L}$.

(c) In the setting when both conjectures make sense, they are related by: $\check{\mathcal{E}}$ is the Langlands transform $\mathcal{L}(\mathcal{O}_{\mathcal{E}})$ of the structure sheaf of the point $\mathcal{E}$in $\mathcal{LS}_{\check{G}}(C)$.

1.1.4. *Remarks.* (0) Some of this is imprecise, for instance a question of the correct category on $\mathcal{LS}_{\check{G}}(C)$.

(1) Part (b) is the original Geometric Langlands conjecture of Drinfeld which is a direct geometrization of a Langlands conjecture in Number Theory. It makes sense over any closed field $\Bbbk$ since it involves only the stack $\mathrm{Bun}_G(C)$, however the original interest was in the case when $\Bbbk$ is the algebraic closure over a finite field since this is relevant for Number Theory. For $G = GL_n$ and $\Bbbk = \mathbb{C}$ this conjecture has been proved by Gaitsgory.

(2) Claim (a) is a lift of (b) to a categorical statement As we will see, if we know or believe (a) then the claim (b) roughly says that the Langlands transform exchanges tensoring of coherent sheaves on $\mathcal{LS}_{\check{G}}(C)$ and convolution of D-modules on $\mathrm{Bun}_G(C)$. (Recall that the usual Fourier transform exchanges multiplication and convolution of functions.)

However, (a) contains much more information since it deals with a finer structure of a category. It is also "more reasonable" in the sense that it involves fewer details. (a) has not been predicted from Number Theory but should contain information of interest for Number Theory. (For instance the isomorphism of K-groups of categories?)

(3) The more general *ramified* version deals with local systems on a punctured curve, i.e., local systems on C with singularities at finitely many points. For this generalization one modifies $\mathrm{Bun}_G(C)$ by adding parabolic structures at singular points. Not much is known in mathematical literature, though there are some results for $\mathbb{P}^1$. Recent contribution from physics (Gukov-Witten) seems significant.

1.1.5. *Dependence on the field* $\Bbbk$. In the case $\Bbbk = \mathbb{C}$ that we will be interested in all these spaces are complex manifolds (stacks) and physicists find that they are a part of a larger differential geometric picture which lives in real manifolds.

For general $\Bbbk$, while the stack $\mathrm{Bun}_G(C)$ is defined over $\Bbbk$, the moduli $\mathcal{LS}_{\check{G}}(C)$ of l-adic local systems is a stack over $\overline{\mathbb{Q}_l}$. Also, D-modules on $\mathrm{Bun}_G(C)$ have to be replaced by perverse l-adic sheaves on $\mathrm{Bun}_G(C)$. So, the conjectured equivalence concerns two triangulated categories which are $\overline{\mathbb{Q}_l}$-linear.

1.1.6. Of course, we need to explain the meaning of Hecke eigen-D-modules which we call simply *Hecke eigensheaves*. Contrary to history, we will start by "guessing" (b) from the formal structure of (a), and then we will make these ideas precise by going through the abelian case $G = G_m$ which is a reformulation of standard facts about Jacobian varieties (this is of course the historical development from abelian to nonabelian).

1.2. **The categorical meaning of Hecke operators.** Hecke eigensheaf property turns out to be quite reasonable – just the Fourier transform of the most obvious description of points in space, i.e., of the characterization of structure sheaves of points.

1.2.1. *Eigenvalue property of points.* Conjecture (a), says that we have two equivalent settings $\mathrm{Bun}_G(C)$ and $\mathcal{LS}_{\check{G}}(C)$. A $\check{G}$-local system $\mathcal{E}$ is a point of $\mathcal{LS}_{\check{G}}(C)$ and since we know something about $D^b[\mathcal{C}oh(\mathcal{LS}_{\check{G}}(C)]$ the question arises how does one describe a point a of a scheme X in terms of the category $D^b[\mathcal{C}oh(X)]$? The simplest answer is that the structure sheaf of a point behaves simple under tensoring with a vector bundle V – we get a multiple of $\mathcal{O}_a$ with multiplicity given by the fiber of V at a :

$$V\otimes_{\mathcal{O}_X}\mathcal{O}_a \ \cong\ V_a\otimes_{\Bbbk}\mathcal{O}_a.$$

We can say that $\mathcal{O}_a$ is a common eigensheaf for operators $V\otimes_{\mathcal{O}_X}-$, where V goes through all vector bundles on X. Notice that our "operators" are really functors and the corresponding "eigenvalue" is a functor from vector bundles to vector spaces – the fiber of at a.

1.2.2. *Tautological vector bundles on $\mathcal{LS}_{\check{G}}(C)$.* To see what the above philosophy means for a point $\mathcal{E}$ in $\mathcal{LS}_{\check{G}}(C)$ we need to know some vector bundles on $\mathcal{LS}_{\check{G}}(C)$. Over $\mathcal{LS}_{\check{G}}(C)\times C$ there is a tautological $\check{G}$-torsor $\mathcal{P}$, for any local system $\mathcal{E}=(P,\nabla)$ the restriction of $\mathcal{P}$ to $\mathcal{E}\times C$ is the underlying $\check{G}$-torsor P. Of course, $\mathcal{P}$ is really the *tautological family of local systems* over $\mathcal{LS}_{\check{G}}(C)$ and it carries a connection in the direction of C, i.e., a relative connection for the projection to $\mathcal{LS}_{\check{G}}(C)$.

Now, any representation V of $\check{G}$ defines a vector bundle $\widetilde{V}\overset{\text{def}}{=}\mathcal{P}\times_{\check{G}}V$ on $\mathcal{LS}_{\check{G}}(C)\times C$. In particular, any point $a\in C$ defines a vector bundle V_a on $\mathcal{LS}_{\check{G}}(C)$ – the restriction of $\widetilde{V}$ to $\mathcal{LS}_{\check{G}}(C)\times a$. Therefore, $\mathcal{O}_{\mathcal{E}}$ is an eigenvector for the *tensoring operators* $\mathcal{W}_{a,V}\overset{\text{def}}{=}V_a\otimes-$:

$$V_a\otimes_{\mathcal{O}_{\mathcal{LS}_{\check{G}}(C)}}\mathcal{O}_{\mathcal{E}}\ \cong\ (V_a)_{\mathcal{E}}\otimes_{\Bbbk}\mathcal{O}_{\mathcal{E}}\ ,$$

and the eigenvalue for the tensoring operator $\mathcal{W}_{a,V}$ is the fiber (a vector space)

$$(V_a)_{\mathcal{E}}=\ P_a\times_{\check{G}}V=\ \mathcal{E}_a\times_{\check{G}}V,$$

where $\mathcal{E}_a=\mathcal{P}_a$ is the fiber of the local system $\mathcal{E}$ at $a\in C$.

Remarks. (1) Instead of using one point of C at a time, one could also write a single equation over $\mathcal{LS}_{\check{G}}(C)\times C$

$$\widetilde{V}\otimes_{\mathcal{O}_{\mathcal{LS}_{\check{G}}(C)\times C}}\mathcal{O}_{\mathcal{E}\times C}\ \cong\ \widetilde{V}|_{\mathcal{E}\times C}\otimes_{\mathcal{O}_C}\mathcal{O}_{\mathcal{E}\times C}$$

for each V *inRep*$(\check{G})$. Here C is the parameter space for operators and the eigenvalue is therefore a vector bundle on C (actually a local system!)

$$(\widetilde{V})_{\mathcal{E}\times C}=\ \mathcal{E}\times_{\check{G}}V.$$

(2) Here we use of all representations V of $\check{G}$, i.e., all conversions of the $\check{G}$-local system $\mathcal{E}$ into a local system of vector spaces, in order to be able to tensor. So, the meaning should really be (in a correct categorical framework)

$$\mathcal{P} \times_{\check{G}} \mathcal{O}_{\mathcal{E}\times C} \ \cong\ \mathcal{E} \times_{\check{G}} \mathcal{O}_{\mathcal{E}\times C},$$

so the eigenvalue is really the local system $\mathcal{E}$ itself.

1.2.3. *Algebra of tensoring operators.* Notice that at each point $a \in C$ our operators form a copy of the tensor category $(Rep(\check{G}), \otimes_{\Bbbk})$ since clearly

(1) $\mathcal{W}_{a,U\oplus V} = \mathcal{W}_{a,U} \oplus \mathcal{W}_{a,V}$ and
(2) $\mathcal{W}_{a,U\otimes V} = \mathcal{W}_{a,U} \circ \mathcal{W}_{a,V}$.

Observation (1) implies that it is sufficient to describe these operators for irreducible V's.

1.2.4. *Langlands transform as a Fourier transform.* Since we are dealing here with "dual" groups, we may hope that the transform $\mathcal{L}$ is *somewhat* like the Fourier transform. The usual F-transform exchanges multiplication and convolution of functions. So, one may hope that the Langlands transform will exchange tensoring which is alike multiplication (it is done pointwise!), with something like convolution. On the coherent side the structure sheaf $\mathcal{O}_{\mathcal{E}}$ is an eigenvector of *tensoring operators* $V_a \otimes_{\mathcal{O}} -$ indexed by all $V \in Rep(\check{G})$ and points $a \in C$, and the eigenvalue is $\mathcal{E}$. Then on the D-module side $\mathcal{L}_{\mathcal{E}}$ should be characterized as a common eigenvector with eigenvalue $\mathcal{E}$ for a family of operators that are (roughly) convolutions with Langlands transforms of tautological bundles. These new operators will be called *Hecke operators* $\mathcal{H}_a(V)$ (once they are defined), they will also be parameterized by the same data V, a.

Next, we check that all of this really works when $G = G_m$ and we use this case to guess what Hecke operators should be for general G.

1.3. **Geometric Langlands conjectures for line bundles.** Here, we notice that for $G = G_m$ Langlands conjectures amount to classical facts about the Jacobian, We start by noticing that in this case the relevant moduli are classical objects. Then the categorical equivalence is proclaimed to be a version of the Fourier-Mukai transform based on the self-duality of the Jacobian.

Now, we come to our real subject – the meaning of Hecke operators for $G = G_m$. As the passage from a curve to its Jacobian is a form of *linearization* where a geometric object is encoded in an abelian group, the same is true for a passage from a local system $\mathcal{E}$ on C to the corresponding D-module $\check{\mathcal{E}}$ on the Jacobian (actually just a local system on the Jacobian). So, the characterizing property for $\check{\mathcal{E}}$ is its compatibility with the group structure on the Jacobian. It turns out that we can reduce this property to its minimal formulation as compatibility of $\check{\mathcal{E}}$ with the modification operators on line bundles $H_a :\ L \mapsto L(a)$ defined at each point $a \in C$. The upshot is that the characterizing property can be stated as an eigenvector property for the operators $T_a,\ a \in C$, which act on local systems on $\mathrm{Bun}_G(C)$ by the pull-backs with respect to the modification operators H_a. These are the Hecke operators $T_a \stackrel{\text{def}}{=} H_a^*$.

1.3.1. *Moduli of line bundles.* A torsor P for $G = G_m$ is the same as a line bundle L. Here P is obtained from L by removing the zero section and $L = P\times_{G_m}\Bbbk$. The standard version of the moduli of line bundles is the Jacobian J_C which is an algebraic ind-group (its connected components are abelian varieties). However, each G_m-torsor (or line bundle) on C has automorphism group G_m and therefore the stack moduli $\mathrm{Bun}_{G_m}(C)$ is slightly more subtle:

$$\mathrm{Bun}_{G_m}(C) \;\cong\; J_C/G_m = \; J_C\times\frac{pt}{G_m},$$

for the trivial action of G_m on J_C.

Let me mention some of the familiar aspects of Jacobians:

(1) The connected components J_C^n of the Jacobian are indexed by the degree $n \in \mathbb{Z}$ of the line bundle.
(2) (Abel-Jacobi maps.) For $n \geq 0$ there is a canonical map $C^n \to J_C^n$ by $(a_1, ..., a_n)\mapsto \mathcal{O}(a_1 + \cdots + a_n)$. It factors through the n^{th} symmetric power of the curve $C^{(n)} \stackrel{\text{def}}{=} (C^n)//S_n$ (the categorical quotient), which is the moduli of effective divisors of degree n. This yields the *Abel-Jacobi maps* AJ_n :

$$C^n \xrightarrow{q_n} C^{(n)} \xrightarrow{AJ^n} J_C^n, \quad AJ_n(D) = \mathcal{O}(D).$$

(3) (Abelian group.) Jacobian is the commutative group ind-scheme freely generated by the algebraic variety C. What this means is that there is a canonical map $AJ^1 : C \to J_C^1$ and it is universal among all maps of C into commutative ind-algebraic groups, i.e., any such map $C \to A$ factors uniquely through J_C.[(2)]
(4) (Self-duality.)[(3)] J_C^0 is an abelian variety which is *canonically* self-dual.

1.3.2. *Moduli of invertible local systems.* Local systems for $\check{G} = G_m$ are the same as *invertible local systems*, so we denote the moduli by $\mathcal{ILS}(C) \cong \mathcal{LS}_{G_m}(C)$. There is a forgetful map

$$\mathcal{ILS}(C) \xrightarrow{\mathcal{F}} J_C,$$

since an invertible local system is a pair $\mathcal{E} = (L, \nabla)$ of a line bundle L and a connection ∇ on L, The map is really $\mathcal{ILS}(C) \xrightarrow{\mathcal{F}} J_C^0$ because a line bundle that carries a connection necessary has degree 0, The fiber of this forgetful map at the trivial line bundle , i.e., all connections on the trivial line bundle is natural identified with the vector space $\Gamma(C, \omega_C)$ of global 1-forms. More generally, the connections on any line bundle $L \in J_C^0$ are a torsor

[2] The correspondence $C\mapsto J_C$ is the origin of Grothendieck's idea of motives – one should associate to algebraic varieties commutative ind-groups which contain most of the information about the variety by Torelli theorems.

[3] This is one many formulations of Class Field Theory.

for $\Gamma(C,\Omega_C)$ since any two connections differ by a 1-form. So $\mathcal{ILS}(C)$ is a torsor for the vector bundle $J_C^0{\times}\Gamma(C,\omega_C)$ over J_C^0.

The (co)tangent bundles over J_C are trivial vector bundles since J_C is a group, So $TJ_C \cong J_C{\times}H^1(C,\mathcal{O})$ because the tangent space at 1 is the linearization $H^1(C,\mathcal{O})$ of $J_C = H^1(C,G_m)$. By Serre duality $T^*J_C = J_C{\times}\Gamma(C,\Omega_C)$. Therefore, $\mathcal{ILS}(C) \to J_C^0$ is a torsor for $T^*J_C^0$.

Notice that adding a connection ∇ to a line bundle L reduces the the automorphism group G_m of L.

1.3.3. *Categorical Geometric Langlands conjecture for* G_m. The derived category of coherent sheaves on $\mathrm{Bun}_G(C) = J_C/G_m$ is the derived category of G_m-equivariant, i.e., graded, coherent sheaves on the Jacobian :

$$D^b[\mathcal{C}oh(\mathrm{Bun}_{G_m}(C))] = D^b[\mathcal{C}oh_{G_m}(J_C)].$$

On the other hand, we have D-modules on $\mathcal{LS}_{G_m}(C) = \mathcal{ILS}(C)$, which is a torsor for $T^*J_C^0$.

So, the categorical part of the Langlands conjecture for G_m says that

Theorem. There is canonical equivalence

$$D^b[\mathcal{C}oh_{G_m}(J_C)] \;\cong\; D^b[mod(\mathcal{D}_{\mathcal{ILS}(C)})].$$

“ *Proof.* ” The underlying geometric content here is that the abelian variety J_C is canonically self-dual. Then the equivalence above can be see to be a “twisted” version of the Fourier-Mukai equivalences. In local systems the twist comes from replacing the group $T^*J_C^0$ by its torsor $\mathcal{ILS}$. The corresponding twist on the side of G-bundles is given by replacing coherent sheaves with D-modules. This twisted version is then just a matter of standard group theory.

Historically, Fourier-Mukai equivalence for dual abelian varieties has been a prototype for *Laumon's Fourier transform of 1-motives*, and the equivalence above is essentially a consequence of Laumon's point of view. The best treatment is in [Polishchuk-Rothstein].

1.3.4. *Correspondence of invertible local systems on the curve and on its Jacobian.* For $G = G_m$, the geometric Langlands correspondence should assign to each invertible local system $\mathcal{E}$ on C a D-module $\mathcal{L}_\mathcal{E}$ on $\mathrm{Bun}_{G_m}(C) = J_C{\times}pt/G_m$ which is an “eigensheaf for Hecke operators”. It will turn out that in this case D-module $\mathcal{L}_\mathcal{E}$ is again an invertible local system on J_C and the whole picture – a correspondence of invertible local systems on C and on J_C – is embedded in the standard theory of Jacobians.

To find the mechanism of the correspondence we start with the case $\Bbbk = \mathbb{C}$ where we can use the complex (i.e., transcendental) methods. Then we will see a geometric proofs valid for general $\Bbbk$.

1.3.5. *Characterization of the correspondence* $\mathcal{E}\mapsto\check{\mathcal{E}}$. Over C it is well known that $\pi_1(J_C^n)$ is $H_1(C,\mathbb{Z})$, i.e., the maximal abelian quotient of $\pi_1(C,\mathbb{Z})$. Invertible local systems are just the one dimensional representations of fundamental groups, so for each $n\in\mathbb{Z}$ there is a bijection of invertible local systems $\mathcal{E}$ on C and invertible local systems on J_C^n. For $n=0$ we denote it $\mathcal{E}\mapsto\mathcal{L}_{\mathcal{E}}^0$.

The remaining discrepancy is that there are more invertible local systems on J_C then on C simply because of the disconnectedness of J_C. From this point of view, a natural correspondence $\mathcal{E}\mapsto\mathcal{L}_{\mathcal{E}}$ would follow from any canonical extension procedure $\mathcal{L}^0\mapsto\mathcal{L}$ of invertible local systems $\mathcal{L}^0$ from J_C^0 to J_C.

Question. What should be a characterizing property of such procedure?

The basic property of the Jacobian is that (i) it is an ind-algebraic group, (ii) which is freely generated by C. So,

- From the point of view of (i), the characterizing property of the extension should be the compatibility with the group structure of J_C.
- (ii) Moreover, from the point of view of (ii) we can play with the formulation of this property by sometimes replacing J_C with just C.

The first approach is simpler but is is the second one that will generalize into the notion of Hecke operators for all groups G.

1.3.6. *Compatibility with the group structure.* We want to extend any invertible local system $\mathcal{L}^0$ on J_C^0 to a local system $\mathcal{L}$ on J_C, which is compatible with the multiplication m on J_C. However, this actually requires that any invertible local system on J_C^0 be compatible with the multiplication on J_C^0. Fortunately, this is true in the following sense. For the multiplication $m^0: J_C^0\times J_C^0\to J_C^0$ map one has a natural isomorphism

$$(m^0)^*\mathcal{L}^0 \cong \mathcal{L}^0\boxtimes\mathcal{L}^0,$$

of local systems on $J_C^0\times J_C^0$. In terms of fibers of the local system at points $L.M, L\otimes M\in J_C^0$ this means a family of isomorphisms (continuous in L,M)

$$\mathcal{L}_{L\otimes M}^0 \cong \mathcal{L}_L^0\otimes\mathcal{L}_M^0.$$

For a proof recall that J_C^0 is a quotient of a vector space by a lattice $H_0(C,\mathbb{Z})$ and therefore the map of fundamental groups induced by m^0 is the addition in $H_1(C,\mathbb{Z})$. This gives the required isomorphism of local systems on $J_C^0\times J_C^0$.

Now it is clear what the characterization should be:

- (i) The canonical extension $\mathcal{L}$ of an invertible local system on J_C^0 to J_C should have the property
$$m^*\mathcal{L} \cong \mathcal{L}\boxtimes\mathcal{L}.$$

- (ii) As C generates J_C, this is equivalent to an isomorphism on $C\times J_C$
$$(C\times J_C \xrightarrow{m} J_C)^*\mathcal{L} \ \cong\ \mathcal{L}|_C\boxtimes\mathcal{L}.$$

In (ii) we think of C as a subvariety of J^1_C via the map
$$AJ^1 : C\hookrightarrow J^1_C, \quad AJ^1(a) = \mathcal{O}(a),\ a \in C.$$
So, (ii) is a family of isomorphisms of fibers
$$\mathcal{L}_{L(a)} \cong\ \mathcal{E}_a\otimes_{\Bbbk}\mathcal{L}_L, \qquad L \in J_C,\ a \in C.$$

1.3.7. *Hecke property.* This is just the explicit form of the above characterization (ii). We start with an invertible local system $\mathcal{E}$ on C and consider the corresponding invertible local system $\mathcal{L}^0_{\mathcal{E}}$ on J^0_C. The canonical embedding $AJ_1 : C\hookrightarrow J^1_C$ induces $\pi_1(C)^{\mathrm{ab}}\xrightarrow{\cong}\pi_1(J^1_C)$. Therefore, if we use a choice of some $L \in J^1_C$ to construct an isomorphism $J^0_C\xrightarrow{\cong}J^1_C$, $M\mapsto M\otimes L$, we get a noncanonical embedding $C\hookrightarrow J^0_C$ which again induces $\pi_1(C)^{\mathrm{ab}}\xrightarrow{\cong}\pi_1(J^0_C)$. This means that the corresponding restriction of $\mathcal{L}^0_{\mathcal{E}}$ to C is isomorphic to $\mathcal{E}$.

So the "simplified" form (ii) of the compatibility condition says
$$(C\times J_C \xrightarrow{m} J_C)^*\mathcal{L} \ \cong\ \mathcal{E}\boxtimes\mathcal{L}, \quad \text{i.e.,} \quad \mathcal{L}_{L(a)} \cong \mathcal{E}_a\otimes\mathcal{L}_L, \quad L \in J_C,\ a \in C.$$

We will restate this in the following terminology :

(1) Each point $a \in C$ defines a *Hecke modification map* $H_a : J_C \to J_C$, $L\mapsto L(a)$, hence a *Hecke operator* $T_a = H^*_a$ on local systems on J_C, which is just the pull-back of local systems under H_a.
(2) We require the local system $\mathcal{L}$ corresponding to $\mathcal{E}$ to be an eigenvector for each of the Hecke operators H_a, with eigenvalue $\mathcal{E}_a$:
$$H^*_a\mathcal{L} \cong \mathcal{E}_a\otimes_{\Bbbk}\mathcal{L}, \qquad a \in C.$$
(3) More precisely, pointwise Hecke modifications fit into a global Hecke modification map $H : C\times J_C \to J_C$, and Hecke operator H^* that takes local systems on J_C to local systems on $C\times J_C$. We require, $\mathcal{L}$ to be an eigenvector for H^* with eigenvalue $\mathcal{E}$:
$$H^*\mathcal{L} \cong \mathcal{E}\boxtimes_{\Bbbk}\mathcal{L}.$$

1.3.8. *Geometric (Unramified) Class Field Theory of Deligne.* This is the above correspondence $\mathcal{E}\mapsto\mathcal{L}_E = \mathcal{L}$. We will now give a proof over any closed field $\Bbbk$ though our thinking so far has been transcendental and therefore only made sense for $\Bbbk = \mathbb{C}$.

The importance of general $\Bbbk$ is for the relation with Number Theory.[4] A technical ingredient when we deal with general $\Bbbk$ is that D-modules do not quite work as well, so it

[4]Actually Number theory would prefer a setting more general then a field!

is standard to use a different notion of local systems on an algebraic variety over a closed field – the *l-adic local systems in etale topology.*[5]

The algebraic construction is given by properties of Abel-Jacobi maps:

Theorem. [Deligne] Let C be a connected smooth projective curve over a closed field $\Bbbk$. For each invertible local system $\mathcal{E}$ on C there is a unique invertible local system $\mathcal{L}$ on J_C which is a Hecke eigensheaf with eigenvalue $\mathcal{E}$, i.e.,

$$(J_C \xrightarrow{H_a:L\mapsto L(a)} J_C)^* \mathcal{L} \ \cong \ \mathcal{E}_a\otimes \mathcal{L}, \quad a\in C.$$

More precisely, one requires that the above isomorphisms are continuous in $a\in C$, i.e., that $H^*\mathcal{L}\cong \mathcal{E}\boxtimes\mathcal{L}$.

Proof. In one direction, $\mathcal{E}$ is the pull-back of $\mathcal{L}$ under the embedding $AJ^1: C\hookrightarrow J^1_C$.

In the opposite direction one constructs from an invertible local system $\mathcal{E}$ on C, a family of invertible local systems $\mathcal{E}^n$ on J^n_C by

(1) On each C^n one has invertible local system $\mathcal{E}^{\boxtimes n}=\mathcal{E}\boxtimes\cdots\boxtimes\mathcal{E}$.
(2) Its "S_n-symmetric part" is an invertible local system $\mathcal{E}^{(n)}$ on $C^{(n)}$ constructed by[6]

$$\mathcal{E}^{(n)} \ \stackrel{\text{def}}{=} [(q_n)_*\mathcal{E}^{\boxtimes n}]^{S_n}.$$

(3) Now, for $n>>0$ (say $n\geq 2g-2$), map AJ^n is by Abel-Jacobi theorem a projective bundle. Since $\pi_1(\mathbb{P}^N)=0$, this implies that $\mathcal{E}^{(n)}$ is a pull-back of a local system on J^n_C which we call $\mathcal{L}^n_{\mathcal{E}}$.
(4) The family of local systems $\mathcal{L}^n_{\mathcal{E}}$ constructed above have the compatibility with multiplication property

$$(J^p_C\times J^q_C \xrightarrow{m} J^{p+q}_C)^*\mathcal{L}^{p+q}_{\mathcal{E}} \ \cong \ \mathcal{L}^p_{\mathcal{E}}\boxtimes\mathcal{L}^q_{\mathcal{E}}$$

and this implies that this family extends uniquely to a local system $\mathcal{L}_{\mathcal{E}}$ on all of J_C which is compatible with multiplication.
(5) From construction, for $n>>0$ we have

$$\mathcal{L}^{n+1}_{L(a)} \ \cong \ \mathcal{L}^n_{L(a)}\otimes_{\Bbbk}\mathcal{E}_a, \quad L\in J^n_C,\ a\in C.$$

This implies that the whole $\mathcal{L}_{\mathcal{E}}$ has this property.

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[5]One can also use D-modules in positive characteristic. The known version of this idea yields a seemingly correct, but more shallow, Langlands correspondence [Bezrukavnikov-Braverman].

[6]The crucial point of this construction is that the symmetric part of $\mathcal{E}^{\boxtimes n}$ is again an invertible local system. If $\mathcal{E}$ is a local system of rank $r>1$ then $\mathcal{E}^{(n)}$ is a sheaf which is only generically a local system – off the diagonal divisor. However, this is still the basis of Drinfeld's conjectural construction of unramified automorphic sheaves for GL_2. This approach was developed by Laumon into a conjectural construction for all $GL(r)$ and Gaitsgory verified Laumon's conjecture when $\Bbbk=\mathbb{C}$.

From this construction of $\mathcal{L}_\mathcal{E}$ the composition $\mathcal{E}\mapsto\mathcal{L}_\mathcal{E}\mapsto\mathcal{L}_\mathcal{E}|C$ is identity. To see that the other composition $\mathcal{L}\mapsto \mathcal{E} = \mathcal{L}|_C\mapsto \mathcal{L}_\mathcal{E}$ is also identity, one observes that the Hecke property of $\mathcal{L}$ implies that for points $a_i \in C$

$$\mathcal{L}_{\mathcal{O}(a_1+\cdots+a_n)} \cong \mathcal{E}_{a_1}\otimes\cdots\otimes\mathcal{E}_{a_n}\otimes\mathcal{L}_\mathcal{O} = \mathcal{E}^{[n]}_{(a_1,\ldots,a_n)}\otimes\mathcal{L}_\mathcal{O} \cong \mathcal{E}^{(n)}_{a_1+\cdots+a_n}\otimes\mathcal{L}_\mathcal{O},$$

and for $n >> 0$ this is the definition of

$$(\mathcal{L}^n_\mathcal{E})_{\mathcal{O}(a_1+\cdots+a_n)} \otimes_\Bbbk \mathcal{L}_\mathcal{O}.$$

So, for $n >> 0$ we get natural isomorphisms $\mathcal{L} \cong \mathcal{L}_\mathcal{E}$ on J^n_C, and then the Hecke property for both $\mathcal{L}$ and $\mathcal{L}_\mathcal{E}$ extends this isomorphism to all of J_C.

1.3.9. *Geometric Langlands correspondence for G_m.* To see the Deligne theorem as a case of the Langlands correspondence formalism, let us match the above operators T_a with the expected parameterization of Hecke operators. As the tensoring operators $\mathcal{W}_{a,V}$ on $\mathcal{LS}_{\check{G}}(C)$ are parameterized by pairs of $a \in C,\ V \in Rep(\check{G})$, the correct Hecke operators should have the same parameterization $T_{a,V}\ a \in C,\ V \in Rep(\check{G})$. Moreover, one should have

(1) $T_{a,U\oplus V} = T_{a,U}\oplus T_{a,V}$ and
(2) $T_{a,U\otimes V} = T_{a,U}\circ T_{a,V}$,

since this is true for tensoring operators. Observation (1) implies that it is sufficient to describe these operators for irreducible V's.

For $G = G_m,\ Irr(G) = \{\chi_n;\ n \in \mathbb{Z}\}$ where $\chi_n(z) = z^n$. Since $\chi_{p+q} = \chi_p\otimes\chi_q$, by observation (2) we only need to know T_{a,χ_1} and this is the above operator T_a the pull back of local systems under the map $H_a(L) = L(a),\ L \in J_C$. The more general Hecke operators do not give anything new since $T_{a,\chi_n} = (T_{a,\chi_1})^n = (T_a)^n$, this is the pull back under the modification map $H_{n\cdot a}(L) \overset{\text{def}}{=} L(n\cdot a)$.

1.4. **Hecke modifications and the loop Grassmannian.** Let us summarize our expectations for Hecke operators:

- (i) Hecke operators should have parameterization $T_{a,V}$ with $a \in C,\ V \in Rep(\check{G})$.
- (ii) They should satisfy

  $$T_{a,U\oplus V} = T_{a,U}\oplus T_{a,V} \quad\text{and}\quad T_{a,U\otimes V} = T_{a,U}\circ T_{a,V}.$$

  So, the basic operators $T_{a,V}$ are parameterized by a point $a \in C$ and $V \in Irr(\check{G})$.
- (iii) $T_{a,V}$ should be given by modifications of G-torsors at a which are of "type V".

This subsection examines the modifications of "type V" at a point a.

We start with the Hecke stack $\mathcal{H}$ of all modifications of a G-torsor at a point of the curve. It is a correspondence over $\mathrm{Bun}_G(C)$, so any substack gives an operator on D-modules on

$\mathrm{Bun}_G(C)$. More generally, any D-module K on $\mathcal{H}$ gives an integral operator with kernel K:

$$I_K(\mathcal{F}) \stackrel{\text{def}}{=} q_\star[K \otimes_{\mathcal{O}_\mathcal{H}} p^\star -]$$

where we denote by $p, q : \mathcal{H}\mathrm{Bun}_G(C)$ the canonical projections and by $p^\star, q_\star$ the operations of a pull-back or direct image for D-modules. This formalism will produce the Hecke operators.

The restriction $\mathcal{H}_a$ of the Hecke stack to modifications at a fixed point $a \in C$ will turn out to be equivalent to a simpler object, the so called *loop Grassmannian* $\mathcal{G}_a$ which appears in various parts of mathematics. For instance, from the group-theoretic point of view $\mathcal{G}_a$ is a partial flag variety of a loop group. One can also think of $\mathcal{G}_a$ as the simplest nonabelian cohomology, hence the simplest nonabelian motive. The fact that Hecke operators at each point $a \in C$ should form an "algebra" (actually a tensor category), will correspond to a certain (obvious) algebraic structure on $\mathcal{H}$ and each $\mathcal{G}_a$.[7]

Loop Grassmannian is the object that we are really interested in. It the next we will see that it has a stratification by $Irr(\check{G})$ and this will make sense of "Hecke modifications of type V" for $V \in Irr(\check{G})$.

1.4.1. *Hecke stack.* Any line bundle L has a canonical modification $L(a)$ at a point a, however this is not true for vector bundles. The effect will be that the Hecke modification maps $L \mapsto L(a)$ on line bundles will be replaced by *Hecke correspondences* for general G.

We define the space $\mathcal{H}_{a,P}$ of all *modifications of a G-torsor P at a point a* as the moduli of all pairs (Q, ι) of a G-torsor Q and an isomorphism $\iota : P \xrightarrow{\cong} Q$ on $C - a$, i.e., off the point a. For instance, for a line bundle L this space $\mathcal{H}_{a,L}$ is a discrete set of all $L(n{\cdot}a)$, $n \in \mathbb{Z}$.

More globally, we define the Hecke moduli $\mathcal{H}$ as the moduli of quadruples (P, Q, a, ι) of two torsors P, Q on C, a point $a \in C$ and an identification $\iota : P \xrightarrow{\cong} Q$ off the point a. The fiber at a given point $a \in C$ is the union of all modification spaces at a

$$\boldsymbol{\mathcal{H}}_a \stackrel{\text{def}}{=} \cup_{P \in \mathrm{Bun}_G(C)} \ \mathcal{H}_{a,P}.$$

The map $\mathcal{H} \to \mathrm{Bun}_G(C)^2$ makes the Hecke stack a correspondence over the space $\mathrm{Bun}_G(C)$, which for each fixed $a \in C$ is a loop Grassmannian bundle over the first copy of $\mathrm{Bun}_G(C)$.

1.4.2. *Locality.* Now notice that the construction $\mathcal{H}_{a,P}$ is local in C – since we allow only modifications at a, the moduli will not change if we replace C by any neighborhood U of a and P by the restriction to U. Actually, the same is true for the formal neighborhood $\widehat{a}$ of a in C. Since any P trivializes on $\widehat{a}$, this reduces understanding of $\mathcal{H}_{a,P}$ to the case when P is the trivial G-torsor $\boldsymbol{G} = C \times G$.

[7] $\mathcal{H}_a$ is the "action groupoid" of a "group" $\mathcal{G}_a$.

1.4.3. *Loop Grassmannians.* We define the *loop Grassmannian* (or "*affine Grassmannian*") for $(C, a, \check{G})$ as the moduli $\mathcal{G}_a \stackrel{\text{def}}{=} \mathcal{H}_{a,\boldsymbol{G}}$ of all modifications at a of the trivial torsor $\boldsymbol{G}$. So, this is the moduli of pairs (Q, ι) where Q is a G-torsor on C and ι is the trivialization (equivalently, a section) of Q off a.

1.4.4. *Cohomological interpretation.* First, the moduli of G-torsors $\mathrm{Bun}_G(C)$ on C is the first nonabelian cohomology $H^1(C, G)$. We can think of it as the prototype of a nonabelian motive – the case $G = G_m$ gives the prototype J_C of abelian motives.

The additional information of a trivialization off a means that we are looking at cohomology classes supported at a. So, the loop Grassmannian is the first *local cohomology* at a

$$\mathcal{G}_a = H^1_a(C, G).$$

(One can also view it as compactly supported cohomology $H^1_c(\widehat{a}, G)$ on the local curve $\widehat{a}$.) This is therefore a local version of $\mathrm{Bun}_G(C)$, hence the simplest nonabelian motive. It packs a number of structures which can be summarized by

Theorem. Loop Grassmannian is fascinating.

Question. What is the meaning of the second local cohomology of G at a point of a surface?

1.4.5. *Principle: "Odd cohomologies are hyperkähler ".* The above expression for $\mathcal{G}_a$ uses the cohomology in algebraic geometry. If we work over the complex numbers then we think of $\mathcal{G}_a$ as $H^{1,0}_a(C, G)$. The total De Rham local cohomology then turns out be the cotangent bundle

$$H^1_{DR,a}(C, G) \;\cong\; T^*\mathcal{G}_a.$$

This is the local version (near a) of the moduli $T^*\mathrm{Bun}_G(C)$ of Higgs pairs.

Recall that the odd abelian De Rham cohomology of a Kaehler manifold is naturally a hyperkähler manifold [Simpson]. This principle remains true for the first nonabelian cohomology. Indeed, the global Higgs bundle moduli $T^*\mathrm{Bun}_G(C) = H^1_{DR}(C, G)$ is hyperkähler (result of Hitchin), and the local Higgs moduli $T^*\mathcal{G}$ is also hyperkähler (result of Donaldson).

Moreover, the appearance of $T^*\mathrm{Bun}_G(C)$ in the Beilinson-Drinfeld and Kapustin-Witten approaches to the geometric Langlands correspondence, can now be stated as study of $H^{1,0}$ using the larger context of H^1_{DR}.

Question. The moduli of local systems should be certain Deligne cohomology?

1.4.6. *Loop Grassmannian as a uniformization of $Bun_G(C)$.* We will see that for a fixed G the moduli $\mathrm{Bun}_G(C)$ for all curves C are group quotients of the loop Grassmannian $\mathcal{G}$ for G (and in many ways). For a point $a \in C$ denote $G_{out} \stackrel{\text{def}}{=} Map(C - a, G)$.

Lemma. If G is semisimple then

$$\mathrm{Bun}_G(C) \;\cong\; G_{out}\backslash\mathcal{G}.$$

Proof. If G is semisimple the canonical map $\mathcal{G}_a \ni \;(Q,\iota)\mapsto Q \in \mathrm{Bun}_G(C)$ is surjective since each torsor trivializes off a point. Then it is clearly a G_{out}-torsor since it acts simply transitively on the sections of a torsor over $C-a$.

(1) (G-bundles on $\mathbb{P}^1$.) Let $a=0\in\mathbb{P}^1$ then G_{out} is $G(\mathbb{C}[z^{-1}])$, hence

$$\mathrm{Bun}_G(C) \;\cong\; G(\mathbb{C}[z^{-1}])\backslash\mathcal{G}_a \;\cong\; G(\mathbb{C}[z^{-1}])\backslash G(\mathbb{C}((z))/G(\mathbb{C}[[z]]).$$

Therefore, the set of isomorphism classes of G-bundles on $\mathbb{P}^1$ is the set of $G(\mathbb{C}[z^{-1}])$-orbits in $\mathcal{G}_a$. We will see that it is indexed the same as the set of $G(\mathcal{O})$-orbits in $\mathcal{G}_a$, i.e., by $W\backslash X_*(T) \;\cong\; Irr(\check{G})$.

1.4.7. *Group theoretic view of the loop Grassmannian.* At a point $a\in C$ denote by $\mathcal{O}$ and $\mathcal{K}$ the functions on the formal neighborhood $\widehat{a}$ and on the punctured formal neighborhood $\widetilde{a}$. In terms of any local parameter z at a these are the formal power series $\mathcal{O}=\Bbbk[[z]]$ and the formal Laurent series $\mathcal{K}=\Bbbk((z))$. We call $G(\mathcal{K})$ the loop group and $G(\mathcal{O})$ the disc group at a.

Lemma. $\mathcal{G}_a \cong\; G(\mathcal{K})/G(\mathcal{O})$.

Proof. Let us enlarge $\mathcal{G}_a$ to a moduli $\widetilde{\mathcal{G}}_a$ of triples (Q,ι,η) where Q is a G-torsor on C and ι and η are respectively trivializations of Q off a, i.e., on $C-a$ and near a, i.e., on $\widehat{a}$. In other words we consider G-torsors on C glued from trivial torsors on $C-a$ and on $\widehat{a}$. These are given by transition functions on the intersection $\widetilde{a}$, i.e., by $\widetilde{\mathcal{G}}_a \cong Map(\widetilde{a},G)=G(\mathcal{K})$. Since $G(\mathcal{O})$ acts simply transitively on all choices of η, we have $\mathcal{G}=\widetilde{\mathcal{G}}_a/G(\mathcal{O})=G(\mathcal{K})/G(\mathcal{O})$.

1.4.8. *Loop Grassmannian of* GL_n. We say that a rank n *lattice* is an $\mathcal{O}$-form of $\mathcal{K}^n$, i.e., an $\mathcal{O}$-submodule $L\subseteq\mathcal{K}^n$ such that $L\otimes\mathcal{O}\mathcal{K}\xrightarrow{\cong}\mathcal{K}^n$. Equivalently, L has an $\mathcal{O}$-basis which is a $\mathcal{K}$-basis of $\mathbb{K}^n$.

Lemma. For $G=GL_n$, loop Grassmannian $\mathcal{G}_a$ is the moduli of rank n lattices.

Proof. $G(\mathcal{K})=GL_n(\mathcal{K})$ acts transitively on $\mathcal{K}$-bases of $\mathcal{K}^n$, hence also on the set of lattices. However, the stabilizer of the trivial lattice $\mathcal{O}^n$ is clearly $GL_n(\mathcal{O})=G(\mathcal{O})$.

We can see this in a more intrinsic way. $\mathcal{G}_a$ is the moduli of G-torsors, i.e., rank n vector bundles $\mathcal{L}$ on $\widehat{a}$ with a trivialization on $\widetilde{a}$. Taking the global sections this means free rank n $\mathcal{O}$-modules L with the identification $L\otimes_{\mathcal{O}}\mathcal{K}\cong\mathcal{K}^n$.

1.5. **Hecke operators and the stratifications of the loop Grassmannians.**

1.5.1. *Loop groups* $G(\mathcal{K})$. Let $\mathcal{O} = \Bbbk[[z]]\subseteq\mathcal{K} = \Bbbk((z))\supseteq\mathcal{O}_- = \Bbbk[z^{-1}]$. Over the field of constants $\Bbbk$, $\mathcal{O}$ can be viewed as a group scheme while $\mathcal{O}_-$ and $\mathcal{K}$ are group-indschemes. Similarly, the group $G(\mathcal{O})$ can be viewed as the group of $\Bbbk$-points of a group scheme $G_\mathcal{O}$ over $\Bbbk$, while $G(\mathcal{K}) = G_\mathcal{K}(\Bbbk)$ for a group ind-scheme $G_\mathcal{K}$ over $\Bbbk$.

For computational purposes we choose a Cartan subgroup $T\subseteq G$ and two opposite Borel subgroups $B = TN$ and $B^- = TN^-$. Denote by $X^*(T) = \mathrm{Hom}(T, G_m)$ and $X_*(T) = \mathrm{Hom}(G_m, T)$ the groups of *characters* and *cocharacters* of T. As $T \cong {G_m}^n$ and $\mathrm{Hom}(G_m, G_m) \cong \mathbb{Z}$ we have $X^*(T) \cong \mathbb{Z}^n \cong X_*(T)$. However, the natural relation is $X^*(T)\xrightarrow{\cong} \mathrm{Hom}_\mathbb{Z}[X_*(T), \mathbb{Z}]$. Denote by W the Weyl group $N_G(T)/T$.

Loop group $G_\mathcal{K}$ (almost) lies in the class of Kac-Moody groups which has structure theory parallel to the standard structure theory of reductive algebraic groups. More precisely, $G_\mathcal{K}$ is close to a Kac-Moody group $\widehat{G}$. Let $R = G_m$ be the group of rotations of the infinitesimal disc $\mathrm{Spec}(\mathcal{O})$, so R acts on $\mathcal{O}\subseteq\mathcal{K}$ by $s(z) = s^{-1}\cdot z,\ s \in R$, and therefore also on $G_\mathcal{K}$ and $G_\mathcal{O}$. Denote by G_{aff} the semidirect product $G_\mathcal{K} \ltimes R$, then $\widehat{G}$ is a certain central extension $0 \to G_m \to \widehat{G} \to G_{aff} \to 0$. In the end, while the Kac-Moody structure theory applies literally only to $\widehat{G}$, groups $G_\mathcal{K}$ and G_{aff} are close enough so that the notions of Cartan and Borel subgroups, Weyl groups and partial flag varieties make sense for all three groups.

The role of a Cartan subgroup in $G_\mathcal{K}$ is played again by the constant Cartan subgroup $T\subseteq G\subseteq G_\mathcal{K}$, however a more useful version is a Cartan subgroup $T_{aff} = T \ltimes R = T\times R$ in G_{aff}. A new ingredient for loop groups is that they have three basic kinds of Borel subgroups: Iwahori subgroup I, negative Iwahori $I-$ and semi-infinite Iwahori J (actually, these three constructions can be combined to produce more types of Borel subgroups). Here

$$I \overset{\text{def}}{=} (G_\mathcal{O} \xrightarrow{z\mapsto 0} G)^{-1}B, \quad I^- \overset{\text{def}}{=} (G_{\mathcal{O}-} \xrightarrow{z\mapsto\infty} G)^{-1}B^-,$$

while

$$J \overset{\text{def}}{=} \text{the connected component of } B_\mathcal{K} = T_\mathcal{O}N_\mathcal{K}.$$

1.5.2. *Schubert cells in the loop Grassmannian.* Loop Grassmannian $\mathcal{G}$ is a partial flag variety of $G_\mathcal{K}$ and we will consider three kinds of Schubert cells, We start with a map $X_*(T) \to \mathcal{G}$. First, the cocharacters (i.e., the cocharacters of T) embedded into the loop group $X_*(T)\hookrightarrow T(\mathcal{K})$, by restricting cocharacters to the punctured formal neighborhood of ∞ in G_m. Therefore, the composition $X_*(T)\hookrightarrow T(\mathcal{K})\subseteq G(\mathcal{K})\twoheadleftarrow G(\mathcal{K})/G(\mathcal{O}) = \mathcal{G}$ attaches to each cocharacter λ a point of the loop Grassmannian that we denote L_λ. Now,

Lemma. (a) $X_*(T)$ embeds into $\mathcal{G}$,

(b) $X_*(T)\subseteq\mathcal{G}$ is precisely the fixed point set $\mathcal{G}^T = \mathcal{G}^{T_{aff}}$.

(c) Any $\lambda \in X_*(T)$ generates orbits

- $I\cdot L_\lambda$ of finite dimension,

- $I^{-}\cdot L_\lambda$ of finite codimension, and
- $J\cdot L_\lambda$ which is semi-infinite (i.e. of infinite dimension and codimension).

This gives a parameterizations of orbits of each of the groups I, I^{-}, J by $X_*(T)$.

(d) $G_{\mathcal{O}}$-orbits in $\mathcal{G}$ are parameterized by Weyl group orbits in cocharacters by

$$\lambda\mapsto\mathcal{G}_\lambda \overset{\text{def}}{=} G(\mathcal{O})\cdot L_\lambda = \ \cup_{w\in W}\ I\cdot L_{w\lambda}.$$

The analogous claim also holds for $G_{\mathcal{O}^-}$-orbits. □

Proof. The parametrization in (c) follows from (a) and (b) since in a partial flag variety orbits of a Borel are indexed by fixed points of the Cartan. Claim (d) follows from (c) since W is represented in $G\subseteq G_{\mathcal{O}}$.

Corollary. $G_{\mathcal{O}}$-orbits and $G_{\mathcal{O}^-}$-orbits are in a canonical bijection with $Irr(\check{G})$.

Proof. The combinatorial characterization of the dual group $\check{G}$ says that $\check{G}$ should contain a Cartan subgroup $\check{\mathfrak{t}}$ such that

- there is an identification $X_*(T)\cong X^*(\check{\mathfrak{t}})$ and
- this identification identifies the root system $\Delta_{\check{\mathfrak{t}}}(\check{G})$ with the dual of the root system $\Delta_T(G)$.

In particular the Weyl groups are canonically identified.

Now, irreducible representations of $\check{G}$ are parameterized by dominant weights $X^*(\check{\mathfrak{t}})_{dom}\subseteq X^*(\check{\mathfrak{t}})$. However, $X^*(\check{\mathfrak{t}})_{dom}\xrightarrow{\cong}X^*(\check{\mathfrak{t}})/W$ since dominant weight form a section for the W-action. So,

$$Irr(\check{G}) \leftrightarrow X^*(\check{\mathfrak{t}})/W \leftrightarrow X_*(T)/W \leftrightarrow G_{\mathcal{O}}\backslash\mathcal{G}.$$

1.5.3. *Irreducible D-modules associated to $G_{\mathcal{O}}$-orbits.* We have managed to find a stratification of the loop Grassmannian by $Irr(\check{G})$. However, recall that the notion of Hecke operators at a given point of the curve is categorical – these "operators" form a tensor category which is a copy of *Rep*$(\check{G})$ (1.2.3 and the beginning of 1.4). So we really need to create on $\mathcal{G}$ an abelian category equivalent to *Rep*$(\check{G})$, so that its irreducible objects come from $G_{\mathcal{O}}$-orbits.

For this we will have to recall that in the realm of D-modules (or perverse sheaves), it is well known how to associate an irreducible D-module to an irreducible subvariety. This will lead to the following abelian category associated to the stratification of $\mathcal{G}$ by $G_{\mathcal{O}}$-orbits:

$$\mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G}) \overset{\text{def}}{=} \quad G_{\mathcal{O}}\text{-equivariant holonomic D-modules on } \mathcal{G}.$$

Here a D-module M on X is *holonomic* if it is "of the least possible size", i.e., for any M the characteristic variety $Ch(M)\subseteq T^*X$ is coisotropic and holonomicity is the requirement that it be Lagrangian.[8]

For an inclusion of smooth subvarieties $Y\overset{j}{\hookrightarrow}X$, $\mathcal{O}_Y$ is a D-module on Y the direct image $j_\star\mathcal{O}_Y$ is then a D-module on X and

Lemma. (a) If Y is connected then $j_\star\mathcal{O}_Y$ has a unique irreducible submodule L_Y. Actually, $L_Y\hookrightarrow j_\star\mathcal{O}_Y$ is equality on the neighborhood $X-\partial Y$ of Y.

(b) L_Y only depends on the irreducible subvariety $\overline{Y}$.

Examples. (1) If Y is closed then $L_Y = j_\star\mathcal{O}_Y$.

(2) If Y is open then $j_\star\mathcal{O}_Y$ is just the sheaf theoretic direct image $j_*\mathcal{O}_Y$ and L_Y is the submodule $\mathcal{O}_X\subseteq j_\star\mathcal{O}_Y$.

(3) In general, $\overline{Y}$ acquires singularities along ∂Y and then L_Y reflects singularities.

(4) The perverse sheaf of $\mathcal{L}_Y$ is the intersection cohomology complex $IC(\overline{Y})$.

(5) The proof of the lemma is based on the following facts

- (i) holonomic D-modules have finite length,
- (ii) holonomicity is preserved under j_*,
- (iii) $j_*\mathcal{O}_Y$ has no submodules supported on ∂Y.

Here part (ii) is is nontrivial, it uses the notion of b-functions.

1.5.4. *Realization of the category Rep($\check{G}$) on the loop Grassmannians of G.* Abelian category $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ of $G_\mathcal{O}$-equivariant holonomic D-modules on $\mathcal{G}$ contains irreducible objects $\mathcal{I}^{!*}_\lambda \overset{\text{def}}{=} L_{\mathcal{G}_\lambda}$ associated to $G_\mathcal{O}$-orbits $\mathcal{G}_\lambda$ in $\mathcal{G}$. The parametrization is really by W-orbits $W\lambda$ in $X_*(T)$ and we think of it as a parameterization by $Irr(\check{G})$. Actually,

Theorem. [Drinfeld et al] $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ is canonically equivalent to *Rep*($\check{G}$).

Remark. (0) So far, this formulation is not quite meaningful since we have not yet *defined* $\check{G}$, we only characterized its isomorphism class (in the proof of theorem 1.5.4). As we will see, the construction of $\check{G}$ from G is a part of the more precise statement of the theorem.

Proof. The idea of the proof is that a category that is equivalent to a category of representations has to have all *structures and properties* that a category of representations has. Clearly, *Rep*(L) comes with tensoring $U\otimes V$, duality U^* and with a forgetful functor

[8] Recall that $Gr(\mathcal{D}_X)\cong\mathcal{O}_{T^*X}$ so one can think of $\mathcal{D}_X$ as a quantization of T^*X. Then $Ch(M)$ is a classical limit of $M-M$ has an $\mathcal{O}_X$-filtration compatible with with the natural filtration of $\mathcal{D}_X$ and such that $Gr(M)$ is a coherent sheaf on T^*X, then $Ch(M)$ is the support of $Gr(M)$, independent of the choice of a filtration.

$\mathcal{F} : Rep(L) \to \mathrm{Vec}$, and these satisfy a number of properties. The formalism of *Tannakian duality* asserts that any abelian category $(\mathcal{R}, \otimes, -^*, \mathcal{F})$ with analogous structures which satisfy certain list of properties, is equivalent to the category of representations of a group $\mathrm{Aut}(\mathcal{F})$ of automorphisms of the functor $\mathcal{F}$ (called the *fiber functor*).

The fiber functor here is the *De Rham* functor $DR \stackrel{\text{def}}{=} \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\mathcal{G}}}(\mathcal{O}_{\mathcal{G}}, -)$ of taking the flat sections. If we worked with perverse sheaves instead, then the fiber functor would be just the total cohomology $H^*(\mathcal{G}, -)$.

The interesting part is the "tensoring" operation $\otimes$ on $\mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G})$ which we denote by $\mathcal{A}*\mathcal{B}$. It is constructed as a *convolution operation* or as a *fusion operation.*

1.5.5. *Convolution.* Let A be a subgroup of a finite group B then the space of A-bi-invariant functions $\mathcal{H} = \mathbb{C}_{A\tilde{A}}(B) = \mathbb{C}[A\backslash B/A)$ acquires the structure of an algebra often called a *Hecke algebra.* If $A = 1$ this is the group algebra $\mathbb{C}[B]$ with the convolution operation

$$f * g \stackrel{\text{def}}{=} (G{\times}G \xrightarrow{m} G)_*(f{\otimes}g), \quad \text{i.e.,} \quad (f*g)(b) = \sum_{(x,y)\in B^2,\ xy=b} f(x)g(y).$$

For biinvariant functions there are some uninteresting repetitions from the diagonal A-action $a{\circ}(x,y) = (xa^{-1}, ay)$, when we eliminate these we still get an associative algebra structure on $\mathcal{H}$ with $(f*g)(b) = \frac{1}{|A|}\sum_{(x,y)\in B^2,\ xy=b} f(x)g(y)$. To sheafify this, we present it in a maximally geometric way by the diagram

$$A\backslash B/A{\times}A\backslash B/A \xleftarrow{\pi} A\backslash B \times_A B/A \xrightarrow{\overline{m}} A\backslash B/A, \qquad f*g = \overline{\mu}_*\pi^*(f{\otimes}g);$$

where $\times_A$ denotes the *product divided by the above diagonal action of* A and $\overline{m}$ is a factorization of the multiplication m.

Now think of $\mathcal{P}_{\mathbb{G}})(\mathcal{G})$ as the category $\mathcal{P}(G_{\mathcal{O}}\backslash G_{\mathcal{K}}/G_{\mathcal{O}})$ of D-modules (or perverse sheaves) on the stack $G_{\mathcal{O}}\backslash G_{\mathcal{K}}/G_{\mathcal{O}}$. Then the formula above interpreted in the world of D-modules gives an operation on this category

$$\mathcal{F}{\star}\mathcal{G} \stackrel{\text{def}}{=} \overline{\mu}_{\star}\pi^{\star}(\mathcal{F}\boxtimes\mathcal{G}).$$

This straightforward geometrization of Hecke algebra construction provides the "tensoring" operation on $\mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G})$ needed above. It turns out that in this approach the commutativity of the operation is not obvious this prompted Drinfeld to find a second construction for which commutativity is clear.

1.5.6. *Fusion.* Here $A*\mathcal{B}$ is constructed as a certain degeneration of $\mathcal{A}{\otimes}\mathcal{B}/$ For this we define for any curve C ind-scheme $\mathcal{G}_{C^n}$ over C^n with fibers $\mathcal{G}_{(a_1,\ldots,a_n)} \stackrel{\text{def}}{=} H^1_{\{a_1,\ldots,a_n\{}(C,G)$, i.e., over C^n we have a tautological subscheme $\mathcal{T}_n$ of C, i.e., $\mathcal{T}_n{\hookrightarrow}C\tilde{C}^n$, then $\mathcal{G}_{C^n}$ is the first relative cohomology group $H^1_{\mathcal{T}_n/C^n}(C{\times}C^n/C^n, G)$ with support along $\mathcal{T}_n$.

Notice that by our definition the fibers of $\mathcal{G}_{C^2}$ are $\mathcal{G}_{(a,a)} = \mathcal{G}_a$ and $\mathcal{G}_{(a,b)} \cong \mathcal{G}_a \times \mathcal{G}_b$ for $b \neq a$ (multiplicativity of cohomology with respect to the support). None the less,

Lemma. $\mathcal{G}_{C^n}$ is flat over C^n.

Proof. This of course could not happen for finite dimensional schemes since $\mathcal{G}^2$ would be larger then $\mathcal{G}$. However, here we deal with indschemes and then flatness means "inductive system of flat subschemes". This is not difficult to arrange, what happens is that as $a_1, .., a_n$ approach a, the product of closures of orbits $\overline{\mathcal{G}_{(a_1,\lambda_1)}} \times \cdots \overline{\mathcal{G}_{a_n,\lambda_n}} \subseteq \mathcal{G}_{(a_1,\dots,a_n)} \subseteq \mathcal{G}_{C^n}$ approaches the closure of a single but larger orbit $\overline{\mathcal{G}_{a,\lambda_1+\dots+\lambda_n}} \subseteq \mathcal{G}_{(a,\dots,a)} \subseteq \mathcal{G}_{C^n}$. $\square$

Once we choose a formal local parameter z_a at a on C the loop Grassmannian $\mathcal{G}_a$ gets identified with $\mathcal{G} \stackrel{\text{def}}{=} G((z))/G[[z]]$. This gives identification $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G}_a)$ with the standard realization $\mathcal{P}_{G[[z]]}(\mathcal{G})$. This identification turns out to be independent of the choice of z_a because $G_\mathcal{O}$-equivariant sheaves on $\mathcal{G}$ are automatically equivariant under $\mathrm{Aut}(\widehat{a} = \mathrm{Aut}(\mathcal{O})$.[9]

This allows us for any $\mathcal{A}, \mathcal{B} \in \mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ to put $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{G}_a \boxtimes \mathcal{G}_b = \mathcal{G}_{(a,b)}$ for $a \neq b$. Now, in D-modules there is a limiting construction, the *nearby cycle functor* Ψ.

1.5.7. *Such construction also exists in perverse sheaves and in K-theory (called simply specialization map), but not in the realm of coherent sheaves. The effect of this is that there is a convolution on* $D^b[\mathcal{C}oh_{G_\mathcal{O}}(\mathcal{P})]$ *but it is not commutative (recall that only the fusion is manifestly commutative), however it is commutative on the level of K-groups (because in K-theory the convolution can be identified with fusion).* We will use Ψ to produce from a "constant" family of D-modules $\mathcal{A} \boxtimes_{C^2 - \Delta_C} \mathcal{B}$ on the part $\mathcal{G}_{C^2}|_{C^2 - \Delta_C}$ of $\mathcal{G}_{C^2}$ that lies above $C^2 - \Delta_C$, a D-module $\mathcal{A} \ *_C\ \mathbb{B}$ on the restriction pf $\mathcal{G}_{C^2}$ to the diagonal :

$$\mathcal{A} \ *_C\ \mathcal{B} \ \stackrel{\text{def}}{=} \ \Psi(\mathcal{A} \boxtimes \mathcal{B}).$$

This is a "constant" D-module on $\mathcal{G}_{C^2}|_{\Delta_C} \cong \mathcal{G}_C$, so it corresponds to a D-module $\mathcal{A} * \mathcal{B}$ on a single $\mathcal{G}$. In other words,

$$\mathcal{A} * \mathcal{B} \stackrel{\text{def}}{=} \lim_{b \to a} \ \mathcal{A} \boxtimes \mathcal{B}.$$

Remarks. (a) One can think of the limiting process as a collision of perverse sheaves positioned at points a and b. So, if we think of $G_\mathcal{O}$-equivariant irreducible perverse sheaves on $\mathcal{G}$ as "elementary G-particles", i.e., some kind of particle like objects produced from G, then we can say that we have produced the group $\check{G}$ and the category of its representations from the study of collisions of G-particles. This is exactly the logic followed in physics a wile ago when they made sense of collisions of particles by finding a group U whose irreducible representations can be made to correspond to particles, so that the procedure of producing new particles through collisions corresponds to tensoring of representations of U and decomposing the tensor product into irreducible pieces.

[9]Intuitively this is so because $\mathrm{Aut}(\mathcal{O})$ preserves $G_\mathcal{O}$-orbits, but that's not the whole story.

(b) The property of multiplicativity of cohomology with respect to the support, $\mathcal{G}_{a,b} \cong \mathcal{G}_a \times \mathcal{G}_b$ for $b \neq a$ says that the local cohomologies we find at a and b are independent. This sounds like *locality principle* in Quantum Field Theory – measurements at points separated in spacetime (by the light cone), are independent. Actually, our situation is the simplest case of of application of locality, the holomorphic two dimensional conformal field theory.

The mathematical approximation of the the holomorphic two dimensional conformal field theory is given by the notion of *vertex algebras*. Then $\mathcal{G}$ or rather all $\mathcal{G}_{C^n}$ together) form a "geometrized vertex algebra" and one gets standard vertex algebras by taking sections of certain D-modules on $\mathcal{G}$.

(c) The above construction fusion construction of tensor category uses group G as sheaf of coefficients in cohomology. In order to state it without G observe that $\mathcal{G}_a = H^1_a(C, G)$ can be viewed as cohomology for a pair $H^1(S, C - a; G)$ or simply as $H^1(S, G)$ for the space S defined (in homotopy a la Voevodsky) as the quotient of C which contracts $C - a$ to a point. Space S is independent of C and a. Inside of C^2 we consider the union A of curves $C \times a$ and Δ_C and we have a pair $(C^2.C^2 - A)$ which maps to C by pr_1. Let $\mathcal{C} \to C$ be the corresponding quotient of C^2 by contracting the fibers of $C^2 - A$. The central fiber of $\mathcal{C} \to C$ is S while the general fiber is the cowedge $S \vee S$. So, $\mathcal{C}$ is a cobordism of $S \vee S$ and S, so it is something like a (co)group structure on S in cobordism category. This is the structure which produced fusion once we put in the coefficient group G.

Proposition. The fusion construction $*$ coincides with the convolution construction $*$.

Proof. As a bridge between the two constructions one extends the convolution construction to the C^2-setting that houses the fusion construction. Then the main observation is that the fibers in the convolution map grow slowly (it is a "semismall map").

1.5.8. *Hecke operators.* Now we can complete the picture. At each point $a \in C$ the category $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ is a copy of the category $Rep(\check{G})$, and we will see that $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ acts on the category $\mathcal{P}(\mathrm{Bun}_G(C))$ of D-modules on $\mathrm{Bun}_G(C)$ by convolution/fusion. These are the Hecke operators.

The simplest point of view is group theoretical, we can think of

$$\mathcal{P}(\mathrm{Bun}_G(C)) = \mathcal{P}(G_{out} \backslash \mathcal{G}_a) = \mathcal{P}_{G_{out}}(\mathcal{G}_a)$$

as the category of $\mathbb{G}_{out}$-equivariant D-modules on $\mathcal{G}_a$. However, the above constructions of $\mathcal{A} * \mathcal{B}$ do not require that $\mathcal{A}$ be $G_\mathcal{O}$-equivariant (but then the result will not be equivariant either). So either of these constructions yields an action of the tensor category $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ on the category $\mathcal{P}(\mathcal{G})$ of all D-modules on $\mathcal{G}$, and this preserves the natural subcategory $\mathcal{P}_{G_{out}}(\mathcal{G}) \subseteq \mathcal{P}(\mathcal{G})$.

In terms of the fiber functor $\mathcal{F} : \mathcal{P}_{G_\mathcal{O}}(\mathcal{G}) \to \text{Vec},\ \mathcal{F}(M) = \text{RHom}_{\mathcal{D}_\mathcal{G}}(\mathcal{O}_\mathcal{G}, M)$, the Hecke property of $\mathcal{F} \in \mathcal{P}(\text{Bun}_G(C))$ says that for any $\mathcal{A} \in \mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ one has

$$\mathcal{F} * \mathcal{A} \cong\ \mathcal{F}(A) \otimes_\Bbbk \mathcal{F}.$$

It suffices to check this for a collection of $\mathcal{A}$'s that generate the tensor category $\mathcal{P}_\mathcal{G}(G_\mathcal{O})$, for instance the irreducibles $\mathcal{L}_{\mathcal{G}_\lambda}$. For $\lambda \in X^*(\check{\mathfrak{t}})$ denote by $L(\lambda)$ the irreducible representation of $\check{G}$ with extremal weight λ, then the Hecke property of $\mathcal{F} \in \mathcal{P}(\text{Bun}_G(C))$ reduces to

$$\mathcal{F} * \mathcal{L}_{\mathcal{G}_\lambda} \cong\ L(\lambda) \otimes_\Bbbk \mathcal{F}, \quad \lambda \in X_*(T)/W.$$

One can also us just the irreducibles corresponding to the fundamental weights etc.

1.6. **Examples in type A.** For $G = GL_n$, take T to be the diagonal Cartan so canonically $T \cong\ (G_m)^n$ and $X_*(T) \cong \mathbb{Z}^n$. In terms of the standard basis e_i of $\mathcal{K}^n$, the lattice L_λ corresponding to the cocharacter $\lambda \in \mathbb{Z}_n$ is then $\oplus_1^n\ z^{-\lambda_i}\mathcal{O}{\cdot}e_i$. The $G_\mathcal{O}$-orbit $\mathcal{G}_\lambda$ then consists of all lattices L which are in the same position to the trivial lattice $\mathcal{O}^n$ as the lattice L_λ in the sense that the dimensions $\dim[(L + z^p{\cdot}\mathcal{O}^n)/z^p\mathcal{O}^n],\ p \in \mathbb{Z}$, are the same as for L_λ.

In particular, for $\omega_i = (1, ..., 1.0, ..., 0) \in \mathbb{Z}^n - X^*(T)$ with i ones, $1 \le i \le n$, we get $G(\mathcal{O})$-orbits

$$\mathcal{G}_{\omega_i} \overset{\text{def}}{=} \{\text{lattices } L;\ z{\cdot}\mathcal{O}^n \subseteq L \subseteq \mathcal{O}^n \quad \text{and} \quad \dim(L/z{\cdot}\mathcal{O}^n) = i\} \cong\ Gr_i(n).$$

Elementary Hecke modifications for GL_n.

1.7. **Langlands conjectures in number theory.**

1.7.1. *Class Field Theory.* One could say that the the structure number theory number theory attempts to describe is that of the totality of all of finite extensions of $\mathbb{Q}$, i.e., the fields of algebraic numbers. Equivalently this can be viewed as the description of the Galois group $Gal_\mathbb{Q}$ of $\mathbb{Q}$, or of the category of finite dimensional representations of $Gal_\mathbb{Q}$.

The Class Field Theory resolves the abelian part of the problem, i.e., it describes the largest abelian quotient $G_al\mathbb{Q}^{\text{ab}}$ or equivalently its irreducible (i.e., one dimensional) representations.

1.7.2. *Langland's conjectural nonabelian generalization of Class Field Theory.* Langlands conjectures are an attempt to deal with the nonabelian nature of $Gal_\mathbb{Q}$. From the new point of view, the n-dimensional representations ρ of $Gal_\mathbb{Q}$, i.e., group maps $Gal_\mathbb{Q} \to GL_n(\mathbb{C})$, correspond to irreducible *automorphic* representations π_ρ of $GL_n(\mathbb{A}_\mathbb{Q})$.[(10)] The Class Field Theory is indeed the case $n = 1$ of this prediction.

[10]It is actually better to allow ρ to be in a larger class of representations of the so called *Weil group* $W_\mathbb{Q}$ which is an amplified version of $Gal_\mathbb{Q}$ and has $Gal_\mathbb{Q}$ as a quotient.

Here $\mathbb{A}_{\mathbb{Q}}$, the ring of adels, is a large topological ring which is associated to $\mathbb{Q}$. It contains $\mathbb{Q}$ and one can think of it as a "locally compact envelope of $\mathbb{Q}$". This locally compact setting allows us to use analysis based on the Haar measures on locally compact groups.

Qualification "automorphic" means that the representation π_ρ appears as a part of the large representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ on functions on its homogeneous space $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}})$.

1.7.3. *L-functions.* One important feature of the correspondence $\rho\mapsto\pi_\rho$ should be that it preserves the basic numerical invariants – the L-functions

$$L_{\pi_\rho}(s) \ = \ L_\rho(s), \quad s\in\mathbb{C}.$$

L-functions are maybe the closest among mathematical objects to the Feynman integrals in physics. The behavior of automorphic L-functions is understood much better. So, if we would know that the L-functions of Galois representations are also L-functions of some automorphic representations we would obtain deep information on behavior of L-functions of Galois representations. For instance, in this way Fermat conjecture has been reduced to a special case of Langlands conjectures.

1.7.4. *Unramified automorphic representations.* A particularly nice class of automorphic representations are the *unramified* representations. A nice feature of such representations is that they contain a distinguished vector, i.e., a distinguished function which is an eigenfunction of certain *Hecke operators*. Functions of this kind appear in the classical complex analysis as *automorphic functions*. From the adelic point of view automorphic functions are the functions on $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}})/GL_n(\mathbb{O}_{\mathbb{Q}})$ where $GL_n(\mathbb{O}_{\mathbb{Q}})$ is a canonical maximal compact subgroup of $GL_n(\mathbb{A}_{\mathbb{Q}})$. So, the unramified automorphic representations are classified by automorphic functions.

There is also the notion of *unramified representations of* $Gal_{\mathbb{Q}}$. These representations are determined by what they do on particular elements of $Gal_{\mathbb{Q}}$ which act as Frobenius automorphisms at points of $\mathrm{Spec}(Z)$. The unramified part of the global Langlands conjecture then says that to each unramified n-dimensional representation ρ of $Gal_{\mathbb{Q}}$ there corresponds an automorphic function f_ρ with the eigenvalues of Hecke operators prescribed by the values of ρ on Frobenius automorphisms.

1.7.5. *Langlands duality of reductive groups.* However, automorphic functions have classically been studied for various reductive groups, not only for GL_n. Incorporating this Langlands predicted that for any reductive complex group G, group maps $G_{\mathbb{Q}}\to G(\mathbb{C})$ correspond to irreducible *automorphic* representations of $\check{G}(\mathbb{A}_{\mathbb{Q}})$ where $\check{G}$ is another reductive group called the *Langlands dual of* G. The case of GL_n is recovered since for $G=GL_n$ the dual group $\check{G}$ is again isomorphic to GL_n.

The duality operation $G\mapsto\check{G}$ is simple on the combinatorial level (i.e., if we only want to produce an isomorphism class of groups rather then a specific group). Then it is a combination of duality for root systems and duality for lattices (groups isomorphic to some $\mathbb{Z}^n$). However, a functorial construction $G \mapsto \check{G}$ is deeper and more recent, the only known construction (Drinfeld, Ginzburg,...) uses loop groups.

1.7.6. *Attraction.* If we restrict ourselves to $G = GL_n$, we may expect that the conjectures have some reasonable explanations. However, the general case suggests that the theory is much deeper since the relation between G and $\check{G}$ is deep. At this point we see that we are confronted with a mystery of the universe which chooses to present the same facts sometimes in the G form and sometimes in the $\check{G}$ form, though to us the two seem unrelated. It is as if you find yourself in a boat in a sea during a lively storm, and stimulated by the danger to your existence and a wild beauty of the elements, you are pushed into a strong, if irrational, conviction of the unity of the sea and the wind which to the uninitiated may seem to only meet accidentally. And so you ask yourself for a Fourier transform that exchanges the wind and the sea.

The G–$\check{G}$ phenomena turns out to be of general importance. This has indeed been confirmed by appearance of this duality in physics as Montonen-Olive duality generalizing the classical electro-magnetic duality, and in various other string related phenomena such as Seiberg-Witten.

1.7.7. *Curves over finite fields.* The Class Field Theory and the Langlands program go beyond $\mathbb{Q}$ itself and deal with a class of fields F called "global fields of dimension one". These fall into the

(1) *arithmetic case*: all algebraic number fields F,
(2) *geometric case*: transcendence degree one extensions $F/\mathbb{F}$ of finite fields $\mathbb{F}$.

The first case is much deeper, it includes $F = \mathbb{Q}$ and provides a larger setting by studying all finite extensions of $\mathbb{Q}$ at the same time and with the same ideas.

The second case is much simpler since such field is a field of fractions $\mathbb{F}(C)$ of a unique smooth projective algebraic curve C over this finite field $\mathbb{F}$. So, number theoretic objects have geometric interpretations and one can use the methods of algebraic geometry.

From the point of view of the more traditional arithmetic case, the geometric case is a "baby case" – the progress is faster and we gain intuition about the arithmetic case. However, the geometric case is also important as a bridge from number theory to algebraic geometry and other mathematics. It depicts number theory as the deepest study of one dimensional geometry with lessons to be applied to algebraic geometry, differential equations, and more recently to physics.

In the remainder we will now review quickly how the (so called) Geometric Langlands Conjectures arises from the Langlands program in Number Theory applied to the geometric case where $\mathbb{Q}$ is replaced by the field $\mathbb{F}(X)$ of rational functions on a curve X defined over a finite field.

1.7.8. *"Functions-sheaves" dictionary of Grothendieck.* This is the mechanism underlying Drinfeld's "geometric" approach to Langlands conjectures, which produces the conjecture 1.1.3(b). (Also a basis of much else.) The idea is that in a number of settings certain class of sheaves $\mathcal{S}$ behaves as a *categorification* of a certain class S of functions. The two classes happen to be closely related but while functions form a vector space S, the sheaves are one categorical step further since they form a category $\mathcal{S}$.

The basic example is the situation considered by Grothendieck. The geometric object is an algebraic variety X defined over a finite field $\mathbb{F}$ with q elements. One considers the derived category $D^b_{constr}(\overline{X})$ of constructible l-adic sheaves in etale topology on $\overline{X} = X_{\overline{\mathbb{F}}}$ the corresponding variety over the closure $\overline{F}$ of $\mathbb{F}$. The Galois group Gal_F acts on $\overline{X}$ and the fixed points $\overline{X}^{Gal_F}$ are precisely the $\mathbb{F}$-points $X(\mathbb{F})$ of X. A sheaf $\mathcal{F} \in D^b_{constr}(\overline{X})$ is said to be defined over F if it is equivariant under the action of the Galois group $Gal_{\mathbb{F}}$. Since $Gal_{\mathbb{F}}$ is generated by the Frobenius automorphism $Fr_{\mathbb{F}}$, equivariance structure means just an isomorphism $\phi : (Fr_{\mathbb{F}})^* \mathcal{F} \xrightarrow{\cong} \mathcal{F}$ of the Frobenius twisted sheaf and the original sheaf.

Now, any sheaf $\mathcal{F}$ defined over $\mathbb{F}$ defines a function $\chi_{\mathcal{F}}$ on the set $X(\mathbb{F})$, at a Frobenius fixed point $a \in X(F)$, ϕ acts on the stalk $\mathcal{F}_a$ since $(Fr_F^* \mathcal{F})_a = \mathcal{F}_{Fr_{\mathbb{F}}(a)} = \mathcal{F}_a$ and

$$\chi_{\mathcal{F}}(a) \stackrel{\text{def}}{=} Tr[\phi : \mathcal{F}_a \to \mathcal{F}_a].$$

Moreover, One actually gets a sequence of functions where $\chi^n_{\mathcal{F}}$ is defined on $X(\mathbb{F}_n)$ the points of X over the degree n extension $\mathbb{F}_n$ of $\mathbb{F}$, using the trace of ϕ^n.

The main thing about this formalism is that it is functorial under maps $f : X \to Y$, i.e.,

$$\chi_{f^*\mathcal{G}} = (f_{\mathbb{F}})^* \chi_{\mathcal{G}} and \chi_{f_! \mathcal{G}} = (f_{\mathbb{F}})_! \chi_{\mathcal{F}},$$

here $f_{\mathbb{F}}$ denotes the restriction $f : X(\mathbb{F}) \to Y(\mathbb{F})$, f^* and $(f_{\mathbb{F}})^*$ denote the standard pull-backs of sheaves and functions, while $f_!$ is the proper direct image of sheaves and $(f_{\mathbb{F}})_!$ denotes the direct image of functions by taking sums over fibers.

How much does the system of functions record about $\mathcal{F}$? It determines the class of $\mathcal{F}$ in the Grothendieck group of $D^b_{constr}(\overline{X})$ (Chebotarev-Laumon).

While the passage from sheaves on $X_{\mathbb{F}}$ to a system of functions on all $X(\mathbb{F}_n)$ is straightforward, the reverse procedure of *geometrization* is an art. One starts with an interesting function f on $X(\mathbb{F})$ and looks for a sheaf $\mathcal{F}$ such that $\chi_{\mathcal{F}} = f$. The main idea is that if f is built from simpler functions using the functoriality of functions the corresponding functoriality of sheaves should be used to build $\mathcal{F}$.

1.7.9. *Geometrization of the (geometric case of) unramified global Langlands conjectures.* Recall that the unramified Langlands conjecture predicts that to each unramified representation ρ of the Galois group G_F of a global field F of dimension one there should correspond an automorphic function f_ρ on $G(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)$, i.e., an eigenvector for number-theoretic Hecke operators, with eigenvalues prescribed by ρ.

When F is the field of rational functions on a curve X defined over a finite field, Galois group $G(F)$ is roughly the fundamental group of the curve X, more precisely it is a limit of fundamental groups of $X-S$ for finite sets S. So, a representation $\rho : Gal_F\check{G}$ defines a local system on $X-S$ for some finite S, i.e., a "*local system with singularities*" on X. The unramified representations ρ are the ones for which the local system has no singularities.

On the other hand the space $G(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)$ turns out to be the set $\mathrm{Bun}_G(C)(F)$ of F-points of the moduli of F-bundles on C. Now it is clear what should be the "geometrization". Instead of constructing a function f_ρ on $\mathrm{Bun}_G(C)(F)$ we should construct a sheaf $\check{\rho}$ on the stack $\mathrm{Bun}_G(C)$ such that $\chi_{\check{\rho}} = f_\rho$. This specific conjecture 1.1.3(b) was eventually made more transparent by a more abstract conjecture 1.1.3(a).

1.7.10. *Geometric Langlands conjecture vs the original Langlands conjectures.* The initial hope was approximately that: (i) sheaves are a much more subtle setting so one will have to define the problem better, (ii) sheaves are a richer setting with constructions that do not have obvious meaning for functions.

This has not quite worked out. In positive characteristic, i.e., in the geometric case, the proof for GL_n completed by Lafforgues did not use *this* geometrization of Langlands conjectures (but some other ones which is not called "geometric Langlands", due to Drinfeld again).

Another obvious weakness of the geometric conjectures is that they do not even try to address the arithmetic case of the original conjectures. I believe that this will be corrected in near future through a (non existing) formalism of "stochastic algebraic geometry".

However, even in the geometric case the geometric Langlands conjectures reach further then the standard ones. One example is that they are geometric enough to make sense for complex curves, and this lead to relations with representation theory of loop groups and vertex algebras, and to the conjectural S-duality in Quantum Field Theory.

1.8. **Appendix. Dual group $\check{G}$ comes with extra features.** This subsection is an amplification of the above results on the loop Grassmannian which has so far not figured in the Witten approach.

The above construction $G{\mapsto}\check{G}$ is certainly not the only possible. However, this method produces $\check{G}$ with some extra structures.

Ginzburg $\check{G}$ has a canonical choice of a Cartan and Borel subgroup $\check{T}{\subseteq}\check{B}{\subseteq}\check{G}$. On the level of representations, the representations of $\check{G}_\mathbb{Z}$ that we produce from sheaves in $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$ by means of the fiber functor come with a grading and an action of $H^*(\mathcal{G})$.

(1) Moreover, they come with (two) *canonical basis.*
(2) From a complex group G one obtains not only a complex group G but actually the split form $\check{G}_\mathbb{Z}$ of $\check{G}$ over integers. Moreover, for any tensor category $\mathcal{T}$ with properties sufficiently alike the properties of the category of modules over a commutative

ring, the same procedure should give a meaning to $\check{G}$ over $\mathcal{T}$. As proposed by Drinfeld and Lurie, this in particular applies to the category of spectra.

For simplicity in this subsection we often refer to perverse sheaves rather then the D-modules because the fiber functor in this case is just the total cohomology. In particular it associates to the irreducible perverse sheaf $\mathcal{L}_{\mathcal{G}_\lambda}$ corresponding to the orbit $\mathcal{G}_\lambda$ (this is the so called intersection cohomology sheaf $\mathcal{IC}(\overline{\mathcal{G}_\lambda})$ of the closure of the orbit), it associates the total *intersection cohomology* of $\overline{\mathcal{G}_\lambda}$.

1.8.1. GL_n. The basic features of the construction are easily seen in the case of $G = GL_n$. The dual group is again GL_n but the geometric construction says more precisely that if our group G is $GL(V)$ for a complex vector space V, then the dual group $\check{G}$ is naturally $GL(\check{V})$ for the complex vector space $\check{V} = H^*[\mathbb{P}(V), \mathbb{C}]$ which we call the Langlands dual of the vector space V.[11]

Clearly, the topological (actually motivic) realization of $\check{V}$ endows it with a $\mathbb{Z}$-form $\check{V}_{\mathbb{Z}} = H^*[\mathbb{P}(V), \mathbb{Z}]$, so the dual group $\check{G} = GL(\check{V})$ comes equipped with a split $\mathbb{Z}$-form $\check{G}_{\mathbb{Z}} = GL(\check{V}_{\mathbb{Z}})$. Moreover $\check{G}$ has a canonical Cartan subgroup $\check{T}$ (the stabilizer of the grading on cohomology), a canonical Borel subgroup $\check{B}$ (the stabilizer of the filtration given by the grading).

It will turn out that all irreducible representations of $\check{G}$, which we realize as intersection cohomology of algebraic varieties, have canonical bases given by irreducible complex subvarieties. For instance the fundamental representations of $GL(\check{V})$ are $L(i) = H^*(Gr_p(V), \mathbb{C})$, $i = 1, ..., \dim V$, and the basis are given by the Schubert varieties in Grassmannians.

1.8.2. *Subgroups* $\check{T}\subseteq\check{B}\subseteq\check{G}$. For a sheaf $M \in \mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$, group $\check{G}$ acts on the corresponding vector space obtained from the fiber functor

$$\mathcal{F}(M) = \mathrm{RHom}_{\mathcal{D}_\mathcal{G}}(\mathcal{O}_\mathcal{G}, M) \cong \mathrm{Ext}^\bullet_{\mathcal{D}_\mathcal{G}}(\mathcal{O}_\mathcal{G}, M).$$

This gives a $\mathbb{Z}$-grading on $\mathcal{F}(M)$ and also a compatible action of $\mathrm{Ext}_{\mathcal{D}_\mathcal{G}}(\mathcal{O}_\mathcal{G}, \mathcal{O}_\mathcal{G}) \xrightarrow{\cong} H^*(\mathcal{G}, \mathbb{C})$.

In particular, we have an action of the hyperplane section class in H^2 which is clearly nilpotent. It turns out that the hyperplane section acts a regular nilpotent $\check{e}$ in the Lie algebra $\check{\mathfrak{g}}$. Moreover, the action of $H^*(\mathcal{G}, \mathbb{C})$ can then be explained through an isomorphism of $H^*(\mathcal{G}, \mathbb{C})$ with the enveloping algebra of the centralizer $Z_{\check{\mathfrak{g}}}(\check{e})\subseteq\check{\mathfrak{g}}$.

The grading can be viewed as an action of the multiplicative group G_m which turns out to come from an embedding $\check{\rho} : G_M \to \check{G}$. On the Lie algebra level this embedding gives a

[11] $\check{G}$ acts on the fundamental representation which is here realized as the intersection cohomology of the closure of the orbit $\mathcal{G}_{\omega_1}$ corresponding to the first fundamental weight. However, $\mathcal{G}_{\omega_1} \cong \mathbb{P}(V)$ and since the orbit is compact the intersection cohomology reduces to ordinary cohomology $H^*[\mathbb{P}(V), \mathbb{C}]$. This action of $\check{G}$ on $\check{V}$ identifies it with $GL(\check{V})$.

semisimple element $\check{h} \in \check{\mathfrak{g}}$. The compatibility of the action of $H^*(\mathcal{G}, \mathbb{C})$ with the grading now says that $\check{e}, \check{h}$ extends to an sl_2-triple $\check{e}, \check{h}, \check{f}$.

Since a regular nilpotent lies in a unique Borel subalgebra $\check{\mathbf{4}}$, $\check{G}$ comes with a distinguished Borel subgroup $\check{B}$. The semisimple part $\check{h}$ of an sl_2-triple with a regular nilpotent is itself a regular semisimple element, so it lies in a unique Cartan subalgebra $\check{\mathfrak{t}}$. Now, $[\check{h}, \check{e}] = 2\check{e}$ implies that the corresponding subgroup $\check{T}$ lies in $\check{B}$.

1.8.3. *Canonical bases of representations.* Recall that the closures of $G_{\mathcal{O}}$-orbits $\overline{\mathcal{G}_\lambda}$ are special finite dimensional Schubert varieties in $\mathcal{G}$, however to each $\nu \in X_*(T)$ we have also associated a semi-infinite Schubert cell $J{\cdot}L_\nu$.

Lemma. (a) For $M \in \mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G})$, there is a canonical decomposition

$$\mathcal{F}(M) \stackrel{\text{def}}{=} H^*(\mathcal{G}, M) \;\cong\; \oplus_{\nu \in X_*(T)} H^*_c(J{\cdot}L_\nu,\ M) \cong\; \oplus_{\nu \in X_*(T)} H^*_{J\cdot L_\nu}(\mathcal{G},\ M).$$

Actually, the summand $H^*_c(J{\cdot}L_\nu,\ M)$ is the ν-weight space of the Cartan subgroup $\check{\mathfrak{t}}$ in the representation $\mathcal{F}(M)$ of $\check{G}$.

(b) In the case when M is an irreducible object, i.e., the intersection cohomology sheaf $\mathcal{IC}(\overline{\mathcal{G}_\lambda})$ of some $G_{\mathcal{O}}$-orbit, the irreducible components $Irr(\overline{\mathcal{G}_\lambda} \cap J{\cdot}L_\nu)$ of the intersection of $\overline{\mathcal{G}_\lambda}$ with the corresponding semi-infinite Schubert cell $J{\cdot}L_\nu$, give a basis of both the ν-weight space of the corresponding irreducible representation,

Remarks. (0) The existence of canonical ones were introduced by Lusztig. bases of irreducible representations is a deep discovery of Lusztig, He introduced two pairs of "canonical" and "semicanonical" bases, the bases above should be related to Lusztig's semicanonical bases.

(1) Part (a) is often called the *geometric Satake* isomorphism, because it is a categorification of Satake's result on p-adic groups.

(2) This is essentially the only known case where genuine intersection cohomology has a basis of algebraic cycles.

(c) Taking the moment map images of the cycles in $Irr(\overline{\mathcal{G}_\lambda} \cap J{\cdot}L_\nu)$ Anderson formulated a charming combinatorics of irreducible representations in terms of certain polytopes.

1.8.4. *Arithmetic content.* The above construction via Loop Grassmannian actually gives a topological interpretation of representations of a split reductive algebraic group $A_{\Bbbk}$ over an arbitrary Noetherian commutative ring $\Bbbk$ of finite global dimension, say $\mathbb{Z}$ or $\mathbb{F}_p$. In this approach, representations of a reductive group $A_{\Bbbk}$ are constructed using the geometry associated to its Langlands dual group G *over complex numbers*, so we start with G and work towards $A_{\Bbbk} = \check{G}_{\Bbbk}$. In this way, category of $\check{G}_{\Bbbk}$-modules has been "localized", i.e. realized as a category of sheaves. This gives a new setting for the study of modular representations of reductive groups (or even the representations over rings), which uses complex geometry.

We have no way to introduce exotic coefficients in the realm of D-modules, so we now switch technologies and on the loop Grassmannian $\mathcal{G}$ of a complex reductive group G we will consider the abelian category $\mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G},\Bbbk)$ of perverse sheaves with coefficients in modules over a commutative ring $\Bbbk$.

Now notice that over a general coefficient ring there are some subtleties which are swept under a rug for $\Bbbk=\mathbb{C}$. A character of a Cartan subgroup $\check{T}$ of the dual group $\check{G}_{\Bbbk}$, then λ defines three $\check{G}_{\Bbbk}$-modules with extremal weight λ

$$W(\lambda,\Bbbk)\rightarrow L(\lambda,\Bbbk)\rightarrow S(\lambda,\Bbbk).$$

Here the Schurr module $S(\lambda,\Bbbk)$ is constructed geometrically as the global sections of a line bundle on the flag variety of $\check{G}_{\Bbbk}$. The Weyl module $W(\lambda,\Bbbk)$ is traditionally introduce in algebra as a quotient o a Verma module. They are related by $S(\lambda,\Bbbk)== W(-\lambda,\Bbbk)^*$. Finally, $L(\lambda,\Bbbk)$ is the image of the canonical map $W(\lambda,\Bbbk)\rightarrow S(\lambda,\Bbbk)$. When $\Bbbk$ is a field this is the irreducible representation with extremal weight λ. If $\Bbbk$ also has characteristic zero then the three constructions coincide.

Theorem. Let $\Bbbk$ be a commutative ring, noetherian and of finite global dimension.

(a) The abelian category $\mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G},\Bbbk)$ together with the convolution/fusion operation $*$ and a certain duality operation is a Tannakian category with a fiber functor given by the global cohomology $\mathcal{F}= H^*(\mathcal{G},-):\mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G},\Bbbk)\rightarrow\Bbbk-mod$. So it is canonically equivalent to the category of representations of the group scheme $\mathrm{Aut}(\mathcal{F})$.

(b) Group scheme $\mathrm{Aut}(\mathcal{F})$ is the split form $\check{G}_{\Bbbk}$ of the dual group $\check{G}$ over $\Bbbk$.

(c) For any cocharacter λ of T, the corresponding $G_{\mathcal{O}}$-orbit $\mathcal{G}_\lambda\xhookrightarrow{j}\mathcal{G}$ defines three perverse sheaves[12]

$$\mathcal{I}_!(\lambda,\Bbbk)\overset{\text{def}}{=}j^p_!(\Bbbk_{\mathcal{G}_\lambda}[\dim\mathcal{G}_\lambda])\rightarrow\mathcal{I}_{!*}(\lambda,\Bbbk)\overset{\text{def}}{=}j_{!*}(\Bbbk_{\mathcal{G}_\lambda}[\dim\mathcal{G}_\lambda])\rightarrow\mathcal{I}_*(\lambda,\Bbbk)\overset{\text{def}}{=}j^p_*(\Bbbk_{\mathcal{G}_\lambda}[\dim\mathcal{G}_\lambda]). \tag{1}$$

The associated representations of $\check{G}_{\Bbbk}$, i.e., the cohomologies of these perverse sheaves are respectively

$$W(\lambda,\Bbbk)\rightarrow L(\lambda,\Bbbk)\rightarrow S(\lambda,\Bbbk).$$

In particular, $L(\lambda,\Bbbk)$ is realized as the standard ("middle perversity") intersection homology of $\overline{\mathcal{G}_\lambda}$ with $\Bbbk$-coefficients.

Theorem. When $\Bbbk$ is the ring of integers situation simplifies since $\mathcal{I}_!(\lambda,\mathbb{Z})=\mathcal{I}_{!*}(\lambda,\mathbb{Z})$ and its dual is $\mathcal{I}_*(\lambda,\mathbb{Z})$.

[12]Once we are not working over $\mathbb{Q}$ there are several versions of intersection cohomology sheaves corresponding to various perversities.

1.8.5. *Example.* In the remainder of this section we consider the simplest example of the convolution and of the coincidence of intersection homology and algebra over a general commutative ring $\Bbbk$.

Let $G = PGL(V)$ for $V = \mathbb{C}^2$, the dual group $\check{G} = SL(\check{V})$ has a fundamental representation $L(1) = \check{V} = H^*(\mathbb{P}(V), \mathbb{C})$ and $L(1)\otimes L(1)$ is a sum of the trivial and adjoint representation $L(0)\oplus L(2) = \mathbb{C}\oplus sl_2(\mathbb{C})$.

Geometrically, two copies of the representations $L(1)$ are realized as constant sheaves (with a shift [1]) on two copies of $\overline{\mathcal{G}_1} = \mathcal{G}_1 = \mathbb{P}(V) = \mathbb{P}^1$. These $\mathbb{P}^1$'s combine in a non-symmetric way into a $\mathbb{P}^1$-bundle $\tilde{X}$ over $\mathbb{P}^1$ Along the "zero" section $\tilde{X}$ is the tangent bundle to $\mathbb{P}^1$, and along the "infinite" section it is the cotangent bundle.

Next, $X = \overline{\mathcal{G}_2}$ is obtained by blowing down the "infinite" section (a (-2)-line), to get $\tilde{X} \xrightarrow{\pi} X$. Then for $\mathcal{L}(n)$ denoting $\mathcal{I}_{!*}(n\check{\rho}, \mathbb{C})$,

$$\mathcal{L}(1) * \mathcal{L}(1) = \mathbb{C}_{\mathbb{P}^1}[1] * \mathbb{C}_{\mathbb{P}^1}[1] = \pi_*\mathbb{C}_{\tilde{X}}[2] = \mathbb{C}_X[2]\oplus \mathbb{C}_{pt} = \mathcal{L}(2)\oplus \mathcal{L}(0).$$

The map $\tilde{X} \xrightarrow{\pi} X = \overline{\mathcal{G}_2}$ is a compactification of the Springer resolution $T^*(\mathbb{P}^1)\to \mathcal{N}$ of the nilpotent cone $\mathcal{N}\subseteq sl_2(\mathbb{C})$, by adding a line $\mathbb{P}^1$ at infinity. The only singularity of X is at $0 \in \mathcal{N}$ and the link of 0 in $\mathcal{N}$ is $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$. This $\mathbb{Z}_2$ is felt in the intersection homology over a general coefficient ring $\Bbbk$. Denote by $\Bbbk_2$ (the 2-torsion in $\Bbbk$) and $\Bbbk/2\Bbbk$ (the 2-cotorsion), also the corresponding sheaves supported at the point $\mathcal{G}_0 = \{0\}\subseteq \mathcal{N}\subseteq \overline{\mathcal{G}_2}$). Then $\mathcal{I}_!(2, \Bbbk) = \Bbbk_X[2]$ and it is an extension $0\to\Bbbk_2\to\mathcal{I}_!(2, \Bbbk)\to\mathcal{I}_{!*}(2, \Bbbk)\to 0$,, while $\mathcal{I}_*(2, \Bbbk)$ is also an extension $0\to\mathcal{I}_{!*}(2, \Bbbk)\to\mathcal{I}_*(2, \Bbbk)\to\Bbbk/2\Bbbk\to 0$.

Since X is paved by affine spaces $H^*[\mathcal{G}, \mathcal{I}_!(2, \Bbbk)] \cong \Bbbk^3$, this is the Weyl module $W(2, \Bbbk) = sl_2(\Bbbk) = \{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right),\ a, b, c \in \Bbbk\}$. Similarly, $H^*[\mathcal{G}, \mathcal{I}_*(2, \Bbbk)]$ is its dual $S(2, \Bbbk) \cong W(2, \Bbbk)^* \cong \Bbbk^3$. The canonical map $\iota : W(2, \Bbbk)\to S(2, \Bbbk)$ is the simplest invariant form on $sl_2(\Bbbk)$: $\langle x, y\rangle \stackrel{\text{def}}{=} tr(xy)$. Since

$$\langle \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & -a' \end{smallmatrix}\right)\rangle = 2aa' + bc' + b'c,$$

one has $\iota \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) = 2a\alpha + b\beta + c\gamma$, for a certain basis α, β, γ of $S(2, \Bbbk)$. So the kernel and the cokernel of ι are $\Bbbk_2$ and $\Bbbk/2\Bbbk$.

The fusion version of convolution in our example means the degeneration of the smooth cubic $\mathbb{P}^1\times\mathbb{P}^1 = \mathcal{G}_1\times\mathcal{G}_2$ in $\mathbb{P}^3$ to a singular cubic $\overline{\mathcal{G}_2}$.

1.8.6. *Remarks.* This motivic point of view also comes with explicit geometric constructions of the group algebra $\mathcal{O}(\check{G})$ and of the positive part $U\check{\mathfrak{n}}$ of the enveloping algebra $U\check{\mathfrak{g}}$ of the dual Lie $\check{\mathfrak{g}}$.

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