QUANTUM FIELD THEORY COURSE

VERSION 03

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Contents

Date: Spring 2000.

Part 0. Introduction

Text.

- **Classical Mechanics**: V. I. Arnold, Mathematical Methods of Classical Mechanics.
- **Strings**: Eric D'Hooker, String Theory ; http://www.math.ias.edu/QFT/spring/index.html.
- **Conformal field theory formalism** : Krzysztof Gawedzki, Lectures on Conformal Field Theory; http://www.math.ias.edu/QFT/fall/index.html.
- **Additional texts on strings**:
  - E. Kiritsis, Introduction to Superstring Theory, hep-th/9709062.
  - J. Polchinski, String Theory Vols. 1 and 2.

0.1. **Topics.** The desired topics would be

- 0. Classical Mechanics,
- 1. Quantum Mechanics,
- 2. Quantum Field Theory and Conformal Field Theory,
- 3. Strings,
- 4. Superstrings,
- 5. Dualities (T-duality, mirror duality),
- Branes
- Calabi-Yau manifolds
- 6. chiral (holomorphic) CFT via vertex algebras,
- 7. Special CFTs (Malikov-Schechtman-Vaintrob and Borisov, Beilinson)

However, we will only touch on (0),(1),(3) and (4).

0.2. **Usefulness of QFT.** Quantum Field Theory is a framework for thinking about the microscopic structure of the world – the nature and behavior of elementary particles.

Our goal is not the frontiers of physics but the *usefulness of the QFT ideas in mathematics*. Importing ideas from QFT to mathematics was the dominant trend in recent past. However we will only learn the basics of QFT and leave the mathematical applications for some other opportunity.

During a period of superficial relations between math and physics, physicists thought deeply about some specific problems and (unobserved) developed a number of ideas that later proved useful in mathematics.

0.2.1. *Some applications to mathematics.*

- Low dimensional topology
  (1) Witten's Chern-Simons invariants of, 3d manifolds,
  (2) Seiberg-Witten invariants of 4d-manifolds.

- Algebraic Geometry: Mirror symmetry.
- Representation theory: Vertex algebras.
- Computing: Quantum computing.

0.2.2. *Two fundamental methods.* There are two basic formalisms used in physics. We will notice how they propagate from Classical to Quantum Mechanics and Quantum Field Theory (in the example of strings):

(1) Lagrangian approach
Feynman integrals calculate the probability $\langle b, a\rangle$ for the system to pass from the state a to the state b as the the sum of contributions from all possible histories:

$$\int_{\text{all ways } x \text{ to pass from } a \text{ to } b} \qquad \text{probability that } x \text{ will happen}$$

(2) Hamiltonian approach
Deformation quantization of algebras of operators

The *relation* of the two approaches is:

Feynman integrals are matrix coefficients of operators.

Mathematics has difficulties with either method, but the situation with (2) has recently improved due to Kontsevich. In (1) there is no mathematical understanding what the Feynman integrals should really mean[1]. The measures are not known, and if they were the integrals would be likely to diverge, and there are claims that whatever we do our expectations for the precise meaning of Feynman integrals are self-contradictory. One way is to imagine that these integrals are only the visible part of some structure finer then just calculating a number. This is a complete million $ mystery.

0.2.3. *The ability of to write down some "functions" via the Lagrangian formalism.* Physicists can write down some functions on sets $\mathcal{S}$ that are mysterious to mathematicians. Such $\mathcal{S}$ is usually the moduli of something (roughly the set of isomorphism classes of something), such as: all smooth manifolds of a certain dimension, all elliptic curves, all Calabi-Yau manifolds of a certain dimension.

The usefulness of such functions:

- They may distinguish different points in $\mathcal{S}$.
  For instance, if $\mathcal{S}$ is the set of isomorphism classes of knots in $\mathbb{R}^3$, if we know a function Z on $\mathcal{S}$ then $Z(K_1) \neq Z(K_2)$ implies that the knots K_1 and K_2 are different. This applies to genetics since the basic structure of different DNA is that they are knotted in a different way.
  We call such functions *invariants* because one often thinks of a knot K in presence of some additional data (a choice of a projection to a plane), then "invariance" means that Z does not depend on these auxiliary choice.

[1]except in some simplest cases: the Gaussian integrals

- Detecting "dualities".
  For instance on the moduli of Calabi-Yau manifolds of dimension n, one can define two invariants Z_A and Z_B (called the A and B models). It has been observed that for each Calabi-Yau M there seems to exist another Calabi-Yau N such that $Z_A(M) = Z_B(N)$. Then M and N are said to be mirror partners. This was spectacular because (i) features about M that are very deep and difficult to understand are often easy to read off from some simple features of N, (ii) it points out an undreamed of relation between familiar geometric objects.
- Construction of interesting functions.
  In the case of the moduli of elliptic curves physicists produce with ease a wealth of the so called modular functions have been studied deeply by number theorists.

0.2.4. *String duality before string theory: Langlands duality.* The Langlands duality between reductive algebraic groups, $G \mapsto \check{G}$, is a deep mystery in representation theory, number theory and algebraic geometry. With the development of string theory, it appears now that this is a particular manifestation of a string duality ideas conjectured by physicists.

In the remainder we survey the notes.

0.3. **Classical Mechanics.** The evolution of a mechanical system is viewed as a path $x : \mathbb{R} \to \mathcal{C}$ in the space $\mathcal{C}$ of all possible configurations (states) of the system. This path satisfies Newton's equation of motion $a = F/m$, which is a second order differential equation

$$\ddot{x} = F(x, \dot{x}, t).$$

For instance, if the force is conservative and independent of velocity and time, this becomes

$$\ddot{x} = \frac{dV}{dx}$$

for the potential $V(x)$. In particular, the situation is completely deterministic: if one knows position and velocity at one moment then there is only one possible evolution.

The two main approaches to the study of Newton's equation of motion are the *Lagrangian* and *Hamiltonian* formulation. These are two geometric ways to think of our differential equations that uncover more symmetries of the situation.

0.3.1. *Lagrangian approach to Classical Mechanics.* It is also referred to as the path approach (since the main heroes are the paths of possible evolutions of the system) or the Calculus of Variations (for its mathematical underpinning). The main idea is to view Newton's equation as a technical manifestation of the principle

physical system evolves so that a certain quantity $S[x]$ is (locally) minimal among all possible evolutions x.

Here one thinks of the curve x in $\mathcal{C}$ in terms of the velocity curve $(x, \dot{x})$ in the tangent bundle $T\mathcal{C}$ to the configuration space $\mathcal{C}$. Newton's differential equation $\ddot{x} = F(x, \dot{x}, t)$,

i.e., the expression F, is here encoded as a function L on $T\mathcal{C}\times\mathbb{R}$, called the *Lagrangian* of the situation. $L(x,\dot{x},t)$ is typically(?) the difference of the kinetic and potential energy $L = T - V$.

Now the *action* $S[x]$ is the time integral of the Lagrangian:

$$S[x] \stackrel{\text{def}}{=} \int_{t_u i}^{t_f} dt\ L(x(t),\dot{x}(t),t).$$

Related mathematical ideas. Critical points of functions are studied in *Morse Theory* (global structure of spaces), and *Stationary Approximation Method* (oscillating integrals $\int\ e^{iS(x)}\ dx$).

0.3.2. *Hamiltonian approach to Classical Mechanics.* It is also referred to as the *Canonical formalism*, her the word "canonical" indicates the relation with the cotangent bundle (as in "canonical transforms"). We think of it as a next step after the Lagrangian approach which lives in the tangent bundle $T\mathcal{C}$ of the configuration space. The idea is that one replaces $T\mathcal{C}$ with the cotangent bundle $T^*\mathcal{C}$ which has an additional geometric structure, the *Poisson structure*.

The passage from $T\mathcal{C}$ to $T^*\mathcal{C}$ is based on the interpretation of the kinetic energy as a metric g_T on $\mathcal{C}$ (i.e., on the vector bundle $T\mathcal{C}$), and on the ideas of the Legendre transform (from functions on $T\mathcal{C}$ to functions on $T^*\mathcal{C}$ and the Legendre map (the vertical differential of a function on $T\mathcal{C}$ is viewed as a map from $T\mathcal{C}$ to $T^*\mathcal{C}$).

The coordinates on $T^*\mathcal{C}$ are traditionally denoted (q,p) with $q = x$ the position in $\mathcal{C}$ and p the momentum. One replaces the velocity curve $(x,\dot{x})$ in the tangent bundle $T\mathcal{C}$, by the momentum curve $(q,p) = (x,p)$ in the cotangent bundle $T^*\mathcal{C}$, using the identification given by the kinetic energy metric g_T. The Lagrangian function L on $T\mathcal{C}$ is replaced by the Hamiltonian function H on $T^*\mathcal{C}$, here $H = \mathcal{L}(L)$ is the Legendre transform of L. H is typically(?) the total energy of the system $H = T + V$, i.e., kinetic + potential energy.

Now Poisson structure on $T^*\mathcal{C}$ associates to the function H a vector field $\widetilde{H} = \{H,-\}$ on $T^*\mathcal{C}$, and Newton's equation of motion gets reformulated into:

The evolution of any observable $f \in C^\infty(T^*\mathcal{C})$ along the momentum curve $(Q(t),p(t)$ is governed by the Hamiltonian vector field: $\frac{d}{dt}f = \{H,f\} = \widetilde{H}(f)$.

So the system is governed by its energy distribution on $T^*\mathcal{C}$. If one applies this to the coordinate function q_i, p_i we get Hamiltonian equations for the evolution of the momentum curve

$$\dot{q} = -H_p \quad \text{and} \quad \dot{p} = H_q.$$

Related mathematical ideas. The basic geometric feature used in the Hamiltonian approach to classical mechanics is the *Poisson geometry*, or more specially the *Symplectic*

Geometry . The study of the Hamiltonian equations lead to the theory of *Integrable systems* (more formally: "Completely Integrable Systems"). This refers to particularly nice systems of PDEs that are not interesting for applied mathematicians since very few PDEs are completely integrable, but are a staple of physics and mathematics since the PDEs in these worlds are often completely integrable as a reflection of the beautiful organization of the world with many hidden structures and interrelations.

0.4. **Quantum Mechanics.**

0.5. **Quantum Field Theory and Conformal Field Theory.**

0.6. **Strings.**

0.7. **Superstrings.**

0.8. **Dualities. T-duality. Mirror duality.**

0.9. **Branes.** .

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OVERVIEW:			
Theory	Features	Lagrangian formulation	Hamiltonian formulation
Classical	$\mathbb{R}\underline{\overset{x}{\to}}\mathcal{C}$	$[t_i,t_f]\underline{\overset{(x,\dot{x})}{\to}}T\mathcal{C},\ L\in C^\infty(T\mathcal{C})$	$\mathbb{R}\underline{\overset{(q,p)}{\to}}T^*\mathcal{C},$
Mechanics	$\ddot{x}=F(x,\dot{x},t)$	$S\in C^\infty(Map[[t_i,t_f],\mathcal{C}]$	$H=\mathcal{L}(L)\in C^\infty(T^*\mathcal{C})$
	Deterministic	$S=\int_{t_i}^{t_f}\ L(x(t),\dot{x}(t),t)\ dt$	$\dot{q}=-H_p,\ \dot{p}=H_q$, i.e.,
		$d_xS=0$ i.e., $L_x=\frac{d}{dt}L\dot{x}$	$\frac{d}{dt}f=\{H,f\}=\widehat{H}(f)$
Quantum	Stochastic	Amplitude= path integral	Amplitude= matrix coefficient
Mechanics	probability		Hilbert space is $\mathcal{H}=L^2(\mathcal{C})$
	$=\lvert\text{amplitude}\rvert^2$		$f\in\mathcal{O}(T^*\mathcal{C})$ gives $\widehat{f}:\mathcal{H}\to\mathcal{H}$
	i.e., $\langle b,t_f\lvert a,t_i\rangle$	$Z(b,t_f\lvert a,t_i\rangle)=$	$Z(b,t_f\lvert a,t_i\rangle)$
	$=\ \lvert Z(b,t_f\lvert a\rvert t_i\rangle)\rvert^2$	$=\int_{x(t_i)=a}^{x(t_f)=b}\ dx\ e^{i\hbar S[x]}$	$=\ \langle\delta_b\lvert e^{-i(t_f-t_i)\widehat{H}}\rvert\delta_a\rangle$
QFT			
Strings	Q-Mechanics of	$4\pi S[x]=$	Hilbert space $\mathcal{H}$
	loop space $\Lambda(M)$	$\frac{1}{4\pi}\int_0^{2\pi}\ d\theta\ \int_{t_i}^{t_f}\ dt\ \frac{\partial x}{\partial t}^2+\frac{\partial x}{\partial\theta}^2$	with an action of
Tree level	of spacetime M	$Z_\Sigma(b,t_f\lvert a,t_i\rangle)$	Heisenberg Lie algebra
(cylinders)	(QFT of the	$=\int_{\Sigma\overset{x}{\to}M,\ x(t_i)=x_i}Dx\ e^{i\hbar S[x]}$	i.e., operators $\widehat{x}_n$ with
q-corrections	worldsheet	$Z_\Sigma(b,t_f\lvert a,t_i\rangle)$	
(R-surfaces)	surface Σ)	$=\int_\Sigma D\Sigma\int_g\ Dg\frac{1}{d(g)?}\int\ Dx\ e^{i\hbar S[x]}$	$[\widehat{x}_n,\widehat{x}_m]=\delta_{m+n}\cdot 1_{\mathcal{H}}$

Part I. Classical Mechanics

The classical mechanics is governed by Newton's first law $ma = F$ which is mathematically a second order ordinary differential equation. There are two traditional approaches which geometrize the study of this Newton equation, the Lagrange and the Hamiltonian approaches. Both carry over to quantum mechanics, QFT and therefore also to strings.

0. Intro

0.1. Mechanical systems.

0.1.1. *Evolution of a system as a path in the configuration space.* Our basic space $\mathcal{S}$ will be an affine space over certain $\mathbb{R}^p$ (we will usually say that $p = 3$). Its meaning is that these are possible positions of one object. The configuration space for n objects (i.e., n-tuples of points in $\mathcal{S}$) is then $\mathcal{C} = \mathcal{S}^n$. The *evolution* of a system of n points is then a *path in the configuration space*, i.e., a map $x : \mathbb{R} \to \mathcal{C}$ from the *time line* to the configuration space. It is an n-tuple $x = (x_1, ..., x_n)$ with x_i mapping $\mathbb{R}$ to $\mathcal{S}$.

0.1.2. *The nature of physical laws in classical mechanics.* The evolution is governed by the *equation of motion*, which we also call *Newton's equation* . It is of the form

$$\textit{acceleration} = \textit{force/mass}, \text{ i.e., } \quad \ddot{x} = F(x, \dot{x}, t).$$

Notice that asking for the equation to be of this form is essentially the same as asking that

If one knows x and $\dot{x}$ at t_0, then $x(t)$ is determined at any time t.

In particular, this encodes the *Deterministic Nature* of the classical mechanics.

A specific system is now described by a specific force, i.e., a specific differential equation of the form $\ddot{x} = F(x, \dot{x}, t)$. The basic example (globally conservative force) is when the force is a gradient vector field of a potential function

$$m_i \ddot{x}_i = \frac{dV}{dx_i}$$

for the potential energy function $V = V(x)$ on $\mathcal{C}$.

0.2. **Lagrangian and Hamiltonian formulation.** These are two geometric ways to think of our differential equations that uncover more symmetries of the situation.

The *Lagrangian approach* is to study the velocity curve $\dot{x}$ in the tangent bundle $T\mathcal{C}$ to the configuration space $\mathcal{C}$, rather then x itself. The mathematical formalism here is based on the idea that the system evolves in a way which minimizes certain quantity.

The *Hamiltonian formulation* replaces $T\mathcal{C}$ with the cotangent bundle $T^*\mathcal{C}$, and the velocity curve with the momentum curve. The advantage is that any cotangent bundle has an additional geometric structure, the Poisson structure.

1. **Lagrangian approach**

1.0.1. *Physical Laws as criticality equations.* The main idea here is that Newton's equation of motion means that the evolution x of the system develops in such a way that a certain quantity, the *action* $S[x]$ of x, is minimal possible.

In order to construct the action $S[x]$, one starts with a function L on the tangent bundle $T\mathcal{C}$, called the Lagrangian function (or just "Lagrangian"). This is (typically?) the difference between the kinetic and potential energy

$$L \stackrel{\text{def}}{=} T - V.$$

Then the action $S[x]$ is a time integral of the Lagrangian function $L(x, \dot{x})$.

Geometrically, the physical setting is now described by a function L on the tangent bundle $T\mathcal{C}$. So, the physics happens on $T\mathcal{C}$, and therefore the time evolution is naturally thought of as a velocity curve $(x, \dot{x})$ in $T\mathcal{C}$[2]. So, the reformulation of the Newton equation for x in terms of the velocity curve, happens to be the criticality equation, of the "action functional" $S[x]$ on the space of all velocity curves $(x, \dot{x})$ in $T\mathcal{C}$ (i.e., all curves x in $\mathcal{C}$).

1.0.2. *Functionals.* A function on a space of functions is called a functional and one denotes it by $S[x]$ rather then $S(x)$, just to remember that x itself is a function. The differential of a functional S with respect to a function x is called variation and denoted

$$\frac{\delta S}{\delta x} \stackrel{\text{def}}{=} d_x S.$$

1.0.3. *Calculus of variations.* This is the underlying mathematical machinery, i.e., the calculation of the criticality equation (the *Euler-Lagrange equation*)

$$\frac{\delta S}{\delta x} = 0.$$

So, we will calculate the variation $\frac{\delta S}{\delta x}$ of the action functional given as a time integral of a Lagrangian function.

Mathematically, this is the differentiation of a function S on an infinite dimensional manifold, so we review the the differential calculus on such manifolds.

1.1. **Manifolds.** The basic assumption of mathematical physics is that the configuration space is a manifold. So we review the basic manifold notions.

[2]This means that we now replace x by a curve (x, v) in $T\mathcal{C}$ and therefore we view the 2^{nd} order Newton equation $\ddot{x} = F(x, y, t)$ in x as a first order system: $\dot{x} = v,\ \dot{v} = F(x, y, t)$ on $T\mathcal{C}$, i.e., a flow on $T\mathcal{C}$.

1.1.1. *Definition.*

1.1.2. *Smooth maps.*

1.1.3. *(Co)tangent spaces.*

1.1.4. *Vector bundles.* Examples: TM and T^*M.

1.1.5. *The differential* $d_aF = F'(a)$.

1.2. **Differentiation.** The criticality equation for the action functional $S[x]$ asks that the differential d_xS vanishes. Since x varies through an infinite-dimensional manifold of smooth functions, we need the calculus is infinite dimensional setting.

1.2.1. *Normed vector spaces.*

1.2.2. *Derivatives of maps between normed vector spaces.*

1.2.3. *Affine spaces over vector spaces.*

1.2.4. *Comparison of notions of a differential.*

1.3. **Calculus of Variations on an interval.** We are interested in the critical points of a functional $S[x]$ on the space of all paths $x : \mathbb{R} \to \mathcal{C}$ in a space $\mathcal{C}$. We will call $S[x]$ the *action* of the path x. The action functional will be a "time" integral

$$S[x] \stackrel{\text{def}}{=} \int_{t_1}^{t_2} L(x(t), \dot{x}(t), t)\ dt,$$

of a function $L(x, v, t)$ which is a (possibly time dependent) function of the position x in $\mathcal{C}$, and velocity $v \in T_x\mathcal{C}$. So the *Lagrangian* $L(x, v, t)$ is a function on $T\mathcal{C}\times\mathbb{R}$.

Bellow we calculate the criticality equation (Euler-Lagrange equation) of this situation.

We will revisit this in the section III.1 where the maps are allowed to have a higher dimensional source.

1.3.1. *Paths.* Precisely, we are interested in the space $C^\infty(I, M)$ of smooth paths from an interval $I = [\tau_i, \tau_f]$, to a Riemannian manifold M. Here we consider the action of the form

$$S[x] = \int_I d\tau\ L(x, \dot{x}, \tau).$$

For simplicity we will assume M to be a vector space. The parameter τ may be time (but need not). The dot above a letter denotes the differentiation with respect to τ.

1.3.2. *Theorem.* The variation (differential) of the action $S[x] = \int_I d\tau\ L(x(\tau), \dot{x}(\tau), \tau)$, in the direction of $u \in T_x(C^\infty(I,M)) = C^\infty(I,M))$, is

$$\frac{\delta S}{\delta x} u = [L_{\dot{x}}(x,\dot{x},\tau)\ u(\tau)]_{\tau_i}^{\tau_f} \quad + \quad \int_I d\tau\ [L_x(x,\dot{x},\tau) - \frac{d}{d\tau} L_{\dot{x}}(x,\dot{x},\tau)] \cdot u$$

$$= \Delta(L_{\dot{x}} \cdot u) + \int_I d\tau\ (L_x - \frac{d}{d\tau} L_{\dot{x}}) \cdot u.$$

(b) The boundary term does not appear in the following cases:

- (i) If we consider paths with fixed choices of end points.
- (ii) If x is a loop $x: S^1 \to M$ and $L = L(x, \dot{x})$ does not depend on time.

(c) When the boundary term does not appear, the EL-equation (criticality equation) is

$$L_x(x,\dot{x},\tau) = \frac{d}{d\tau} L_{\dot{x}}(x,\dot{x},\tau), \quad i.e., \quad L_x = \frac{d}{dt} L_{\dot{x}}.$$

Proof. (a) In general a tangent vector $u \in T_x(C^\infty(I,M))$ is the section of TM along x, i.e., a section of the pullback x^*TM. When M is a vector space this is the same as $u \in C^\infty(I,M))$. The variation $\frac{\delta S}{\delta x}$ of S at x and with respect to u, means the differential

$$\frac{\delta S}{\delta x} u = (d_x S)u \stackrel{\text{def}}{=} \frac{d}{d\varepsilon}|_{\varepsilon=0}\ S(x+\varepsilon u) = \int_I d\tau\ \frac{d}{d\varepsilon}|_{\varepsilon=0}\ L(x+u, \dot{x}+\dot{u}, \tau)$$

$$= \int_I d\tau\ L_x(x,\dot{x},\tau)u + L_{\dot{x}}(x,\dot{x},\tau)\dot{u}.$$

It remains to use integration by parts for the "dynamic" ingredient $\dot{u}$.

(b) In the case (i) one has $u = 0$ on ∂I. In (ii), everything is periodic with respect to $\tau_f - \tau_i$.

1.3.3. *Pointwise variation* $\frac{\delta S}{\delta x(t)}$. Now we consider the role of the variation with respect to the value of x at t. So, let $t \in I - \partial I$ and $v \in T_{x(t)}M$. We are calculating the variation of $S[x]$ when one moves x by a distributional path $u = \delta_t(\tau) \cdot v$ concentrated at the point $t \in I$, so

$$\frac{\delta S}{\delta x(t)} v \stackrel{\text{def}}{=} \frac{\delta S}{\delta x}\ \delta_t(\tau) \cdot v = (d_x S)\ \delta_t(\tau) \cdot v.$$

Corollary. The pointwise variation is the integral kernel for the variation operator $\frac{\delta S}{\delta x}$

$$\frac{\delta S}{\delta x(t)} = [L_x - \frac{d}{d\tau} L](x(t), \dot{x}(t), t), \quad t \in (t_i, t_f).$$

Proof. When t is not an end point the boundary term vanishes and $(d_x S)\ \delta_t(\tau)v$ is

$$\int_I d\tau\ [L_x(x(\tau),\dot{x}(\tau),\tau) - \frac{d}{d\tau} L_{\dot{x}}(x(\tau),\dot{x}(\tau),\tau)] \cdot \delta_t(\tau)v = [L_x - \frac{d}{d\tau} L](x(t),\dot{x}(t),t) \cdot v.$$

In terms of component functions and the Einstein summation convention this is

$$\frac{\delta S}{\delta x(t)} v \;=\; L_{x^\mu}(x(t),\dot{x}(t),t)\; v^\mu - \frac{d}{d\tau} L_{d\dot{x}^\mu}(x(t),\dot{x}(t),t)\; v^\mu.$$

1.3.4. *Remarks.* (a) The variation with respect to x consists of two parts, due to the appearance of x as a *non-dynamical* variable (i.e., the appearance of the field x itself without derivatives), and as a *dynamical* variable (i.e., the appearance of the derivative $\dot{x}$ of the field). The non-dynamical term of the variation has no time derivative, while the dynamical term has time derivative with a minus (since the dynamic variable causes integration by parts).

(b) In general, i.e., when we have to allow the boundary term, EL-equation consists of two parts: $L_x = \frac{d}{dt} L_{\dot{x}}$ and $L_{\dot{x}}(x(\tau),\dot{x}(\tau),\tau) = 0$ at the ends $\tau \in \partial I = \{\tau_i, \tau_f\}$. (For the first claim we use all u's supported in (τ_i, τ_f) and the continuity at the ends. Then for the second we just vary the values of u at the ends.)

1.3.5. *The case* $S(x) = \int_{t_1}^{t_2} L(x(t))\, dt$.

1.4. **Lagrangian reformulation of Newton's equation.**

1.4.1. *Newton's equation as a flow on* $T\mathcal{C}$. Instead of a curve x into $\mathcal{C}$ one considers the corresponding velocity curve $(x, \dot{x})$ into the tangent bundle $T\mathcal{C}$. The equation on $T\mathcal{C}$ is of the 1st order, so it gives a flow on $T\mathcal{C}$.

1.4.2. *Lemma.* In the standard case of the conservative force, i.e., $m\ddot{x} = -\frac{dV}{dx}$, one can indeed reformulate Newton's equation as the Euler-Lagrange equation.

Proof. The corresponding Lagrangian function $L \in C^\infty(T\mathcal{C})$, is

$$L \stackrel{\text{def}}{=} T - V = \text{ the difference between the kinetic energy and the potential energy.}$$

$$\text{i.e., } L(x(t),\dot{x}(t)) = T(\dot{x}(t)) - V(x(t)) = \tfrac{1}{2} m\dot{x}(t)^2 - V((x(t)).$$

The action is the time integral of the Lagrangian

$$S[x] \stackrel{\text{def}}{=} \int_{\tau_i}^{\tau_f} dt\; L(x(t),\dot{x}(t),t) = \int_{\tau_i}^{\tau_f} dt\;\; \frac{1}{2} m\dot{x}(t)^2 - V((x(t)).$$

Since, $L_x = -\frac{dV}{dx}$ and $L_{\dot{x}} = m\dot{x}$, the criticality equation $L_x = \frac{d}{dt} L_{\dot{x}}$ is indeed

$$-\tfrac{dV}{dx} = -\tfrac{d}{dt} m\dot{x}, \;\text{ i.e., } m\ddot{x} \;=\; \tfrac{dV}{dx}.$$

1.4.3. *Terminology in the presence of a Lagrangian function* L *on* $T\mathcal{C}$.

1.4.4. *Examples of Lagrangians.*

- (a) Mass falling to Earth: $L = \frac{1}{2}m\dot{x}^2 - mgh$.
- (b) Harmonic oscillator : $L = \frac{1}{2}(m\dot{x}^2 - mx^2)$.

2. **Hamiltonian approach**

Also called the "canonical formalism". Here "canonical" indicates the cotangent bundle setting (as in "canonical transforms").

Newton's equation of motion has last been seen in the Lagrangian formalism as the Euler-Lagrange (criticality) equation for the velocity curve in the tangent bundle to the configuration space. We will now reformulate Newton's equation in terms of the momentum curve in the symplectic variety $T^*\mathcal{C}$ (the "phase space"). This "Hamiltonian" equation says that the evolution of the momentum curve is given by the Hamiltonian vector $\widetilde{H}$ field on $T^*\mathcal{C}$, i.e., the vector field associated to the Hamiltonian function H via the symplectic structure on $T^*\mathcal{C}$.

Technically, the passage from $T\mathcal{C}$ to $T^*\mathcal{C}$ uses the ideas of the Legendre map and the Legendre transform, and a natural metric g_T on $T\mathcal{C}$ given by the kinetic energy T. canonical metric The function H is the phase space incarnation of the Lagrangian function L on the tangent bundle $T\mathcal{C}$, explicitly, $H = \mathcal{L}(L)$ is the Legendre transform of L. In the standard case when L is the difference of the kinetic and potential energy, H is the total energy $H = T + V$ of the system.

The canonical momentum p is a $T^*\mathcal{C}$ version of the momentum $m\dot{x} \in T\mathcal{C}$. It is defined as the Legendre map of the Lagrangian L (i.e., the differential of L).

The kinetic energy metric g_T is used to move the velocity curve $(x(t), \dot{x}(t))$ in $T\mathcal{C}$ to the momentum curve $(q, p(t))$ in $T^*\mathcal{C}$.

In terms of the coordinates (q, p) =(position,momentum) on $T^*\mathcal{C}$, the Euler-Lagrange equation $L_x = \frac{d}{dt}L_{\dot{x}}$ is first reformulated as the Hamiltonian equation for the evolution of the momentum curve

$$\dot{p} = -H_q \quad \text{and} \quad \dot{q} = H_p,$$

and this is elegantly written in terms of the Poisson structure on $T^*\mathcal{C}$ as

$$\dot{p} = \{H, q\} \ \& \ \dot{q} = \{H, p\}.$$

This gives a uniform formula for the evolution of any observable $f \in \mathbb{C}^\infty(T^*\mathcal{C})$ along the momentum curve:

$$\dot{f} \ = \ \{H, f\} \ = \ \widetilde{H}f.$$

The classical usefulness of the Hamiltonian formulation was the large group of symmetries (symmetries of Newton's equation are given by the Galillean group).

2.1. **Metrics: linear algebra.**

2.1.1. *Inner products and quadratic forms.*

Lemma. (a) An inner product $(-,-)$ on a vector space V is the same as a positive non-degenerate quadratic form q on V.

(b) It gives an isomorphism of vector spaces $\iota : V \xrightarrow{\cong} V^*, \quad \iota : v \mapsto (v, -)$, which takes any orthonormal bases v_i to the dual bases p_i. Then $q = \sum p_i^2$.

2.1.2. *Metrics on vector bundles.* A metric g on a vector bundle V/X gives an isomorphism $\iota_g : V \xrightarrow{\cong} V^*$.

2.1.3. *Metrics on a manifolds.* A Riemannian manifold (M, g) is a manifold M with a metric g on M, i.e., on the vector bundle TM. On a Riemannian manifold (M, g) one has the notions:

- (a) length of curves,
- (b) distance, i.e., a metric space structure,
- (c) geodesics.

2.2. **The passage to the cotangent vector bundle via the kinetic energy metric.** We will think of kinetic energy as a metric on the configuration space $\mathcal{C}$, i.e., on its tangent bundle $T\mathcal{C}$.

2.2.1. *Metric and kinetic energy.* The formula for the kinetic energy $T = \sum m_i(\dot{x}_i)^2/2$ uses the standard metric on the configuration space $\mathcal{C} \cong \mathbb{R}^{np}$.

However, one can also think of the quadratic form T on the tangent bundle $T\mathcal{C}$ as a natural metric g_T on $\mathcal{C}$

$$g_T(x, \dot{x}) \stackrel{\text{def}}{=} (\dot{x}, \dot{x})_T \stackrel{\text{def}}{=} \sum m_i(\dot{x}_i)^2, \quad g_T(x; \dot{x}, \dot{y}) \stackrel{\text{def}}{=} \sum m_i \, \dot{x}_i \cdot \dot{y}_i.$$

In this way, *mass* influences the *geometry* of the space.

The corresponding isomorphism of vector bundles $i\iota_{g_T} : T\mathcal{C} \xrightarrow{\cong} T^*\mathcal{C}$ is given on $T_x\mathcal{C}$ by

$$\dot{x} \mapsto (\dot{x}, -)_T = (\sum m_i \dot{x}_i, -),$$

for the standard metric on $\mathcal{C} = \mathbb{R}^{3n}$.

2.2.2. *Momentum curve* $(q, p) : \mathbb{R} \to T^*\mathcal{C}$. This is the image of the velocity curve under the identification $g_T : T\mathcal{C} \xrightarrow{\cong} T^*\mathcal{C}$, given by the kinetic energy metric g_T on $\mathcal{C}$. So, $t \mapsto (q(t), p(t))$ where $q(t) = x(t)$ is the position, and $p(t)$ is the canonical momentum

$$p(t) = (\sum m_i \cdot \dot{x}_i(t), -) = \frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t), t).$$

2.3. **Legendre transform.**

2.3.1. *The differential of $f \in C^\infty(V)$, viewed as a Legendre map $f' : V \to V^*$.* For a function f on a manifold M, its differential f' is a section of T^*M. However, if M is a vector space V we can view it as a "Legendre map" $f' : V \to V^*$. (For $f\mathbb{C}^\infty(V)$, the differential at a, $f'(a) \stackrel{\text{def}}{=} d_a f =: T_a(V) \to T_{f(a)}\mathbb{R}$ is a linear function on $T_a v \cong V$, hence $f'(a) \in V^*$.)

This will generalize the map of vector spaces $V \stackrel{\cong}{\to} V^*$ that comes from an bilinear form $(-,-)$ on V (think of it quadratic form q on V).

If V is a vector bundle over a manifold M then a function f on V defines a fiberwise Legendre map $\partial| f : V \to V^*$ is calculated fiber by fiber, i.e., for $m \in M$, this is the partial derivative in the vertical direction $\partial_| f = f|V_m{}' : V_m \to V_m^*$.

Examples. Observe that $f \mapsto f'$ is linear in f. The map f' associated to a linear function $f = p \in V^*$ is a constant function $f' = p$ from V to V^*. The map associated to a constant function is $c' = 0$.

Lemma. (a) For a positive definite quadratic form q, map $(\frac{q}{2})' : V \to V^*$ is the isomorphism of vector spaces given by the corresponding inner product $(-,-)$.

(b) If f is a strictly convex function then f' is a bijection.

Proof. (a) $q = \sum p_i^2$ for some basis p_i of V^* and then $(u,v) = \sum p_i(u) \cdot p_i(v)$. Now, $(\frac{q}{2})'(a) = \sum p_i(a) \cdot p_i = (a, -)$.

(b) is a generalization of (a). When $V = \mathbb{R}$ it is obvious from the graph.

2.3.2. *Legendre transform $\mathcal{L}$ of functions on V to functions on V^*.* Actually, $\mathcal{L}f$ is defined only when the Legendre map $f' : V \to V^*$ is a bijection, since the definition uses the inverse of the Legendre map:

$$(\mathcal{L}f)(p) \stackrel{\text{def}}{=} \langle v, p\rangle - f(v) \mid_{v = (f')^{-1}p} = \langle v, (f')^{-1}p\rangle - f((f')^{-1}p) , \quad p \in V^*.$$

The geometric meaning is in

Lemma. Suppose that f is strictly convex. Then $\mathcal{L}(f)$ is defined and $(\mathcal{L}f)(p)$ is the maximal height of the graph of the linear function p on V above the graph of the convex function f.

Proof. The height at $v \in V$ is $h(v) = \langle v, p\rangle - f(v)$. The height has a unique critical point since $0 = d_v h = p - f'(v)$ means that $v = (f')^{-1}p$. Actually, since f is strictly convex, so is the height function h and therefore h has a (unique) maximum.

2.3.3. *Lemma.* (Properties.) (a) For a positive definite quadratic form q on V,

$$\mathcal{L}(\frac{q}{2}) = \frac{Q}{2},$$

where the positive definite quadratic form Q on V^* is obtained by transporting q to Q by the isomorphism $\iota_q : V \xrightarrow{\cong} V^*$ defined by q.

(b) $\mathcal{L}(f+c) = \mathcal{L}f - c$.

(c) $\mathcal{L}(cf)\ (p) = (\mathcal{L}f)(c^{-1}\cdot p)$.

(d) $\mathcal{L}(f+q)\ (p) = (\mathcal{L}f)(p-q)$ for $q \in V^*$.

(e) If f' is a bijection then so is g' for $g = \mathcal{L}f$.

(f) $\mathcal{L}(\mathcal{L}f) = f$.

Proof. (a) $\iota_q = (\frac{q}{2})'$, or equivalently, $\iota_q(v) = (v, -)$ for the corresponding inner product.

2.3.4. *Question.* For f convex and even, when is the Fourier transform of $e^{-f/2}$ equal to $e^{-\mathcal{L}f/2}$?

2.4. **The canonical momentum.** A choice of a Lagrangian function L on $T\mathcal{C}$ defines a notion of a *canonical* momentum

$$p \overset{\text{def}}{=} \frac{\partial L}{\partial \dot{x}}.$$

So, the canonical momentum p is the variation (change) of the Lagrangian function corresponding to an infinitesimal change of velocity – our dynamical variable. (one also says that $\frac{\partial L}{\partial \dot{x}}$ is the momentum *conjugate to* the position function x.

Formally, the canonical momentum is the Legendre map (see 2.3.1)

$$T\mathcal{C} \xrightarrow{p} T^*\mathcal{C}$$

for the Lagrange function L on $T\mathcal{C}$. (The partial derivative $\frac{\partial L}{\partial \dot{x}}(x, \dot{x})$ is the differential $d_{\dot{x}}(L|T_x\mathcal{C})$ of a function on a vector space $T_x\mathcal{C}$, so it is a point of $T_x^*\mathcal{C}$.)

2.4.1. *Comparison with the naive momentum (when the potential energy does not depend on velocity !)* In this case the canonical momentum is the change of the kinetic energy corresponding to an infinitesimal change of velocity:

$$p \overset{\text{def}}{=} \frac{dL}{d\dot{x}} = \frac{dT}{d\dot{x}} - \frac{dV}{d\dot{x}} = \frac{dT}{d\dot{x}} = \frac{d}{d\dot{x}} \sum m_i(\dot{x}_i, \dot{x}_i)/2 = (\sum m_i \cdot \dot{x}_i, -).$$

Here the canonical momentum (defined from the Lagrangian), coincides with the naive momentum up to the interpretation as a covector. Observe also that in this case, the canonical momentum map $T\mathcal{C} \xrightarrow{p} T^*\mathcal{C}$, is precisely the isomorphism given by the kinetic energy metric g_T on $T\mathcal{C}$.

Lastly, we will denote by "T on $T_x^*\mathcal{C}$", the kinetic energy function transplanted to a function on $T^*\mathcal{C}$ by the above isomorphism.

2.5. **Hamiltonian reformulation of Newton's equation.**

2.5.1. *Hamiltonian function H on $T^*\mathcal{C}$ as the Laplace transform $\mathcal{L}L$ of the Lagrangian function L on $T\mathcal{C}$.* In order to define the Hamiltonian function on $T^*\mathcal{C}$ by

$$H \stackrel{\text{def}}{=} \mathcal{L}L,$$

we need to assume that the restrictions $L|T_x\mathcal{C}$ to any tangent space has invertible Legendre map. In practice $L|T_x\mathcal{C}$ will be strictly convex:

$$L|T_x\mathcal{C} \;=\; \text{a positive definite quadratic form } T \text{ plus a constant } -V(x).$$

2.5.2. *Lemma.* If L is the difference between the kinetic and potential energy then H is the total energy

Proof. Let $x \in \mathcal{C}$ and $p \in T_x^*\mathcal{C}$. Observe that $q = 2T$ is a positive definite quadratic form on $T_x\mathcal{C}$, and denote by Q the positive definite quadratic form on $T_x^*\mathcal{C}$, obtained by transporting q via the isomorphism $V \stackrel{\cong}{\to} V^*$ given by q. then

$$H(x,p) = \mathcal{L}(L)\,(x,p) \stackrel{\text{def}}{=} \mathcal{L}(L|T_x\mathcal{C})\,(p) = \mathcal{L}(\frac{q}{2} - V(x))\,(p) = \frac{Q}{2} + V(x) = \text{“}T \text{ on } T_x^*\mathcal{C}\text{”} + V(x),$$

where we denote by “T”= “T on $T^*\mathcal{C}$” the kinetic energy function transplanted to a function on $T^*\mathcal{C}$ by the same isomorphism (recall that $\dot{x} \mapsto (\sum m_i \dot{x}_i, -)$).

2.5.3. *Hamiltonian equations for the momentum curve.*

Lemma. In terms of the momentum curve $\mu(t) \stackrel{\text{def}}{=} (q(t), p(t)) : \mathbb{R} \to T^*\mathcal{C}$, the Euler-Lagrange equation for the velocity curve becomes

$$\dot{p}(t) = -\partial_1 H(q(t), p(t), t) \qquad \text{and} \qquad \dot{q}(t) = \partial_2 H(q(t), p(t), t).$$

Less precisely,

$$\dot{p} = -H_q \qquad \text{and} \qquad \dot{q} = H_p.$$

[A terrible crime has been committed in the last line since the symbols p and q have two meanings. On the LHS p is a certain function of time, a component of the momentum curve, on the RHS p is a coordinate on the phase space $T^*\mathcal{C}$.]

Proof. [We will only consider the case when $L = T(\dot{x}) - V(x)$.] While the evolution curve $x : \mathbb{R} \to \mathcal{C}$ satisfies Newton‘s equation, the velocity curve $(x, \dot{x}) : \mathbb{R} \to T\mathcal{C}$ satisfies Euler-Lagrange equation and one definition equation

$$\frac{\partial L}{\partial x}(x(t), \dot{x}(t), t) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t), t), \qquad \text{and} \qquad \dot{x}(t) = \frac{dx}{dt}.$$

By the definition of the momentum curve $p(t) \stackrel{\text{def}}{=} \partial_2 L(x(t), \dot{x}(t))$ so EL equation is reformulated as $\dot{p}(t) = \frac{d}{dt}\partial_2 L(x(t), \dot{x}(t)) = \partial_1 L(x(t), \dot{x}(t))$. Now we move that to $T^*\mathcal{C}$

$$\dot{p}(t) = \partial_1 L(x(t), \dot{x}(t)) = \frac{\partial}{\partial x}\,(T(\dot{x}) - V(x))\;|_{x=x(t),\ \dot{x}=\dot{x}(t)} = -\frac{\partial}{\partial x}\,V(x)\;|_{x=x(t),\ \dot{x}=\dot{x}(t)}$$

$$= -\frac{\partial}{\partial x}\ (\text{“}T\text{”}(p) + V(x))\ |_{x=x(t),\ p=p(t)} = -\frac{\partial}{\partial x}\ H(x(t), p(t)).$$

Also,

$$\partial_2 H(q(t), p(t)) = \partial_2(\text{“}T\text{”}(p) - V(x))\ |_{x=x(t),\ p=p(t)} = d_{p(t)}\text{“}T\text{”} = \dot{x}(t).$$

The last equality is the computation of the Lagrangian map for the function $\text{“}T\text{”}|T^*_{x(t)\mathcal{C}}$ at the point $p(t)$. Now recall that we start with a positive definite quadratic form T and that “T” is its Lagrangian transform, i.e., “the dual” positive definite quadratic form on the dual vector space. It remains to observe that the Legendre maps for dual quadratic forms are mutually inverse.

2.6. **Poisson structures.** We will interpret the Hamiltonian equations above in terms of the Poisson structure, a standard geometric structure on the cotangent bundle.

2.6.1. *Lie algebras.* (a) Commutator in an associative algebra. (b) Vector fields. (c) Lie groups.

2.6.2. *Poisson structures.* (a) On a commutative algebra A. (b) On a manifold M.

(c) The canonical Poisson structure on T^*M. If $M = \mathbb{R}^n$ then

$$\{f, g\} \stackrel{\text{def}}{=} \sum_i f_{p_i} g_{q_i} - f_{q_i} g_{p_i},$$

hence

$$\{p_i, -\} = \partial_{q_i}, \quad \{q_i, -\} = -\partial_{p_i} \quad \text{and} \quad \{p_i, q_j\} = \delta_{ij}.$$

(d) Function f on M defines a vector field $\widetilde{f} \stackrel{\text{def}}{=} \{f, -\}$ on M.

2.7. **Canonical formalism.** Using the canonical Poisson structure on $T^*\mathcal{C}$, Hamiltonian equations are rewritten as

$$\dot{p}(t) = \{H, p\}(\mu(t), t) \qquad \text{and} \qquad \dot{q}(t) = \{H, q\}(\mu(t), t),$$

i.e.,

$$\dot{p} = \{H, p\} \quad \text{and} \quad \dot{q}(t) = \{H, q\}.$$

2.7.1. *Observables.* We call $T^*\mathcal{C}$ the phase space (its principal property is the Poisson structure). We call functions on the phase space observables since $\phi \in C^\infty(T^*\mathcal{C})$ can be observed any time t, i.e., it defines a function $\phi(t) \stackrel{\text{def}}{=} \phi(\mu(t))$ of time. The evolution of $\phi(t)$ is always given by the Hamiltonian

$$\dot{\phi}(\mu(t), t) = \{H, \phi\}(\mu(t), t), \quad i.e., \quad \dot{\phi} = \{H, \phi\}.$$

(This is true for the coordinate functions p_i, q_i hence for all polynomials in them etc.)

This is the final formulation of the canonical formalism.

2.7.2. *Hamiltonian vector field.* This is the vector field $\widetilde{H}$ on $T^*\mathcal{C}$ associated to the Hamiltonian function. In terms of this vector field Hamiltonian equations become

$$\dot{\mu}(t)) = \widetilde{H}(\mu(t)),$$

i.e., the solutions are integral curves of the flow defined by the Hamiltonian vector field.

2.7.3. *Harmonic oscillator example.* There $L = T(\dot{x}) - V(x) = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$, hence

$$H = T(\dot{x}) + V(x) = \frac{1}{2m}p^2 + \frac{k}{2}q^2$$

and

$$\widetilde{H}(f) = \{H, f\} = \{\frac{1}{2m}p^2 - \frac{k}{2}q^2, f\} = \frac{1}{2m}2p\{p, f\} + \frac{k}{2}2q\{q, , f\} = \frac{1}{m}p\partial_q f - k{\cdot}q\partial_p f.$$

So, $\widetilde{H} = \frac{1}{m}p\partial_q - k{\cdot}q\partial_p$. If $k = m = 1$ then $\widetilde{H} = p\partial_q - q\partial_p$ and the integral curves are circles. In general they are the ellipses $\frac{1}{m}p^2 + kq^2 = c,\ c \geq 0$.

2.8. **Summary.** We proved equivalence of several differential equations. The balk of the time went into defining the framework for the quantities involved - manifolds, tangent and cotangent bundles. This is equivalent to specifying how these quantities transform under the change of coordinates which is the classical language that is standard in physics.

If our system of points has some constraints, its configuration space (all possible positions) will be a manifold M (say a submanifold of $\mathcal{C} = \mathcal{S}^n$). Still, the Lagrangian formalism applies with L a function on TM.

The original Newton's law $F = ma$ looks differently in different coordinate systems, i.e., it is not invariant under the transformations of space other then the Galillean group (translations, rotations and ?). The Hamiltonian equation (canonical formalism) is invariant under a large class of transformations of $T^*\mathcal{C}$ (rather then only $\mathcal{C}$) – the ones that preserve the Poisson bracket (canonical transformations or symplectomorphisms). Reducing the problem to the simplest form by a choice of a symplectic coordinate change is said to be the strongest method in classical dynamics (Arnold).

2.9. **Extra: Symplectic structures.**

2.9.1. *Differential forms.* A p-form ω on a manifold M is a smooth family $\omega_m,\ m \in M$, with each ω_p an alternating multi-liner form on T_mM in p-variables. Let $\mathcal{A}^p(M)$ be the vector space of p-forms on M, then $\mathcal{A}^0(M) = C^\infty(M)$.

2.9.2. *De Rham differential.* De Rham differential $d = d_p : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ satisfies $d^2 = 0$. For $p = 0$, $d_0 = df$ is the usual differential etc. We say that a p-form ω is closed if it is killed by d_p.

2.9.3. *Symplectic structures.* A symplectic form on M is a closed 2-form ω such that each bilinear form ω_m on T_mM is non-degenerate.

A symplectic form ω on M defines a Poisson structure $\{-,-\}$ on M and is in turn determined by it. So a symplectic structure is a special case of a Poisson structure.

2.9.4. *Cotangent bundles.* On T^*M one has a canonical 1-form η which in local coordinates can be written as $\eta = \sum p_i\ dq_i$. Then $\omega = d\eta = \sum dp_i\ dq_i$ is a canonical symplectic structure that gives the Poisson structure above.

3. Example: Equivalence of three actions for a Free particle

In the simplest possible example of a free particle we will see how three actions are reasonably equivalent.

- (A) In *Newtonian setting* (i) the obvious action is the kinetic energy and (ii) the evolution of the particle is naturally parameterized by time.
- (B) In a *relativistic length formulation* we consider (i) the action will be the length of the trajectory in spacetime and (ii) we add a (physically meaningless) quantity, a choice of a *parameterization* of the trajectory since the time is now not divorced from space so, there is no canonical parameterization
- (C) The *relativistic kinetic formulation* has (i) action quadratic in $\dot{x}$, (ii) another auxiliary field – a metric g on the worldline I.

The *Newtonian kinetic energy action* is physically natural. The nice mathematical features of the *length action* are that it is familiar to mathematicians (geodesics) and makes sense if we increase the worldsheet dimension (volume action). For physics, it passes the obvious *classical* test: the solution of the equation of motion is the same as in the non-relativistic setting (as it should be since a free particle is inertial).

The unpleasant feature of the length action is that the quantization is difficult since $\dot{x}$ appears under a square root (Lagrangian is really the square root of the kinetic Lagrangian!), and this makes quantization difficult. The "relativistic kinetic action" resolves the last problem (it is quadratic in $\dot{x}$), and also allows for the massless particles.

Its formula is hard to justify directly, however it passes a *classical-quantum* test of comparison with the length action: the solutions of the equation of motion are the same and the new action reduces to the length action at the criticality of the auxiliary field.

3.1. **(A) Newtonian setting: the kinetic energy action.** In general, the Lagrangian is the difference of kinetic and potential energy. The meaning of "free particle" is that the potential is absent, so

$$L = T - V = \frac{m}{2}\dot{x}^2,$$

and the action is a time integral of the Lagrangian

$$S[x] = \int_{t_i}^{t_f} dt L(\dot{x}) = \int_{t_i}^{t_f} dt \frac{m}{2}\dot{x}^2.$$

Therefore, the EL-equation $L_x = \frac{d}{dt}L_{\dot{x}}$ now says

$$0 = L_x = \frac{d}{dt}\frac{m}{2}\dot{x}^2 = m\ddot{x}, \quad i.e., \quad \ddot{x} = 0.$$

So, acceleration is zero, velocity is constant: $\dot{x} = v$, and the solutions are the lines

$$x = x_0 + tv.$$

3.2. **(B) Relativistic setting: the spacetime trajectory length action.**

3.2.1. *Relativistic setting.* For a relativistic setting one needs a formulation in which *time is not separated from space.* So the evolution of a particle should be viewed as a curve C in the spacetime M.

For calculations one still needs a parameterization and now we can not parameterize by time anymore. So, the price we pay is the introduction of an additional datum, a choice of a parameterization x of C by an interval I. The choice of parameterization is physically meaningless since the action does not depend on it (the Lagrangian does). Physicists call this *invariance under reparameterizations*, these new symmetries (reparameterizations) precisely eat up the new degree of freedom.

3.2.2. *Length action.* The action will be the length action $S[C] = l(C)$ for the spacetime trajectory C. The idea is that the most natural evolution is the one that minimizes the trajectory length. Notice that on the level of Lagrangians this is the square root of the above kinetic energy action!

This length action is familiar mathematically (solutions are geodesics), but it is more complicated then the kinetic energy action because of the square root.

The action of the trajectory C, calculated in terms of a parameterization x is

$$S[C] \stackrel{\text{def}}{=} S[x] \stackrel{\text{def}}{=} m \int_C dl = m \int_I d\tau \sqrt{-\dot{x}^2} = \int_I d\tau \, L(\dot{x})$$

where $dl = d\tau \sqrt{-\dot{x}^2}$ is the length of a piece of the curve C and the Lagrangian is

$$L(\dot{x}) = m\sqrt{-\dot{x}^2}.$$

3.2.3. *The role of the mass factor in action.* Removing the mass factor from the action would not affect the EL-equation. However, the next lemma shows that this would fix the size of the canonical momentum to be 1.

3.2.4. *Lemma.* (a) The canonical momentum is[3]

$$p = \frac{m\dot{x}}{\sqrt{-\dot{x}^2}}.$$

(b) The EL-equation (equation of motion) is

$$\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{-\dot{x}^2}}\right) = 0.$$

[3]Here, velocity with lower indices means that we view it as a cotangent vector. This and the mass factor are standard, a nonstandard feature is the denominator. It comes from taking the square root of the kinetic energy in the Lagrangian.

This means that $\frac{\dot{x}}{\sqrt{-\dot{x}^2}} = c$ is a constant vector which is necessarily a unit vector. So, all solutions are of the form

$$\dot{x} = \phi(\tau) \cdot c,$$

where c is a unit vector, and ϕ is any function $\phi : I \to \mathbb{R}_{\geq 0}$ (ϕ is the length of $\dot{x}$).

(c) There are two kinds of symmetries of the action

- (1) Spacetime symmetries: the Poincare group $\mathcal{P}$. These are the diffeomorphisms P of the spacetime that preserve the metric G. Such P moves x to $P \circ x$.
- (2) Group $Diff_0(I)$ of orientation preserving reparameterizations of the interval I. Any $\phi \in Diff_0(I)$ changes x to $x \circ \phi^{-1}$.

Proof. (a) The canonical momentum is defined from the Lagrangian $p \stackrel{\text{def}}{=} \frac{\partial}{\partial \dot{x}} L$. Since

$$\frac{\partial}{\partial \dot{x}^\mu} \dot{x}^2 = \frac{\partial}{\partial \dot{x}^\mu} \eta_{\nu'\nu''} \dot{x}^{\nu'} \dot{x}^{\nu''} = \delta_{\mu\nu'} \eta_{\mu\nu''} \dot{x}^{\nu''} + \delta_{\mu\nu''} \eta_{\mu\nu'} \dot{x}^{\nu'} = 2\eta_{\mu\nu} \dot{x}^\nu = 2\dot{x}_\mu,$$

we find that

$$p_\mu = -m \frac{1}{2} \frac{-2\dot{x}_\mu}{\sqrt{-\dot{x}^2}} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}.$$

(b) The EL-equation is $L_x = \frac{d}{dt} L_{\dot{x}} = \frac{d}{dt} p$, i.e., $0 = \frac{d}{dt} p$.

3.2.5. *Remark.* The huge non-uniqueness of the solutions is just the freedom in reparameterizations of the trajectory C. All curves C are given by the simplest parameterizations $\dot{x}_\mu = c$ for a unit vector c. So the curve C is a line in spacetime. This is the same as the solution we found for the non-relativistic motion in space, as so it should be since inertial motion notices no relativistic effects.

In this sense, the length action in spacetime is a generalization of the standard action in space.

3.3. **(C) Relativistic setting: kinetic action with a worldline metric.** We will *improve the relativistic length action* so that

- (i) it is quadratic in $\dot{x}$ (a la the kinetic energy action) and
- (ii) it allows for massless particles.
- (iii) In some sense it is equivalent to the length action.

A quadratic dependence in $\dot{x}$ is much simpler and preferable for path integrals. The *price* will be the introduction of another *auxiliary field – a choice of a Riemannian metric g on the worldline I.* [4] The action is

$$S[x, g] \stackrel{\text{def}}{=} -\frac{1}{2} \int_I dl \quad (\dot{x})^2_{G,g} - m^2.$$

[4] In the preceding construction, on the worldline we used the pull-back metric $g = x^*G$ coming from a Lorentzian metric G on the spacetime, but we effectively used $-x^*G$ which was Riemannian because of the light cone assumption on the motion.

We will assume that $\boldsymbol{m \neq 0}$ throughout, and revisit the massless case at the end (3.3.4).

I do not know how to motivate this formula,[5] but we will check that it has the properties (i-iii). For the notation see

3.3.1. *The coordinate notation.* The metrics g and $g^* = g^{-1}$ are given by the functions

$$g_{\tau\tau} \stackrel{\text{def}}{=} g(\partial_\tau, \partial_\tau) > 0 \quad \text{and} \quad g^{\tau\tau} \stackrel{\text{def}}{=} g^{-1}(d\tau, d\tau) = \frac{1}{g_{\tau\tau}}$$

on I. We also encoded g in terms of the *einbein*

$$e \stackrel{\text{def}}{=} \sqrt{g}_{\tau\tau} = \text{ length of } \partial_\tau.$$

By dl, we denote the length of a piece of the interval I, i.e., $dl = e(\tau) \cdot d\tau$.

The square speed $-(\dot{x})^2_{G,g} \stackrel{\text{def}}{=} -\dot{x} \underset{G,g}{\cdot} \dot{x}$, is the measure of size of the velocity $\dot{x}(\tau) \in \text{Hom}_{\mathbb{R}}[T_\tau(I), T_{x(\tau)}M] = T^*_\tau(I) \otimes T_{x(\tau)}M$, which is given by the inner product $(g^*)_\tau \otimes G_{x(\tau)}$, so by C.1.2,

$$(\dot{x})^2_{G,g} = \frac{\partial x}{\partial \tau} \underset{G}{\cdot} \frac{\partial x}{\partial \tau} \cdot g^{\tau\tau}.$$

So, in coordinates the action is

$$S[x, g] = -\frac{1}{2} \int_I d\tau \ \sqrt{g_{\tau\tau}} \ (g^{\tau\tau}\dot{x}^2 - m^2) = \int_I d\tau \ L(\dot{x}, g),$$

for the Lagrangian

$$L(\dot{x}, g) = -\frac{1}{2} \ [(g^{\tau\tau})^{\frac{1}{2}}\dot{x}^2 - \ (g^{\tau\tau})^{-\frac{1}{2}}m^2].$$

Notice that x is a dynamical variable and g is not.

3.3.2. *Variations with respect to the inverse metric g^{-1} and the path x.* We have two fields so we will get two criticality equations, the vanishing of the variations (differentials) with respect to g^{-1} and x. [6]

Lemma. (a) The criticality equation $\frac{\delta S}{\delta g^{\tau\tau}} = 0$ with respect to the inverse metric, is

$$g^{\tau\tau} = \frac{m^2}{-\dot{x}^2}, \quad i.e., \quad g_{\tau\tau} = \frac{1}{m^2} \cdot (-\dot{x}^2), \quad i.e., \quad g = \frac{x^*(-G)}{m^2}.$$

It determines the metric g in terms of x – up to a mass factor, g is the x-pull-back of the metric $-G$ on the spacetime.

(b) The criticality equation with respect to the path x: $\frac{\delta S}{\delta g^{\tau\tau}} = 0$, is the requirement that $(g^{\tau\tau})^{\frac{1}{2}}\dot{x}$ is a constant vector, i.e., τ-velocity is a multiple of some vector c

$$\dot{x} = g_{\tau\tau}{}^{\frac{1}{2}} \cdot c.$$

[5] In particular, how do length and mass have the same units?

[6] One uses g^{-1} so that the variation $\frac{\delta S}{\delta g^{\tau\tau}}$ is proportional to g.

Proof. (a) Since the metric g^{-1} is a non-dynamical field

$$\frac{\delta S}{\delta g^{\tau\tau}} = L_{g^{\tau\tau}} = -\frac{1}{2}\,[\frac{1}{2}(g^{\tau\tau})^{-\frac{1}{2}}\dot{x}^2 - (-\frac{1}{2})(g^{\tau\tau})^{-\frac{3}{2}}m^2] = -\frac{1}{4}(g^{\tau\tau})^{-\frac{3}{2}}\,[g^{\tau\tau}\dot{x}^2 + m^2].$$

(b) Since x is a purely dynamical field

$$\frac{\delta S}{\delta \dot{x}} = \frac{\partial}{\partial\tau}L_{\dot{x}}(\dot{x},g) = -\frac{1}{2}\,\frac{\partial}{\partial\tau}\frac{\partial}{\partial\dot{x}}[(g^{\tau\tau})^{\frac{1}{2}}\dot{x}^2 - (g^{\tau\tau})^{-\frac{1}{2}}m^2] = -\frac{1}{2}\,\frac{\partial}{\partial\tau}\,[(g^{\tau\tau})^{\frac{1}{2}}\cdot 2\dot{x}].$$

3.3.3. *The relation to the length action.* We will now check that the passage from the length action to $S[x,g]$ is an example of the philosophy 3.4.1 of adding an auxiliary field to the action, while keeping the physics essentially equivalent.

Lemma. (a) (*Reduction at criticality.*) At the criticality for the metric g^{-1}, the new action reduces to the length action

$$S[x,g]\big|_{g=\frac{-x^*G}{m^2}} = \int_I d\tau\; m\sqrt{-\dot{x}^2} = S_{length}[x].$$

(b) (*Classical equivalence.*) The criticality equations of the new action $S[g,x]$ in both the metric g and the path x, together amount to the criticality in x of the length action and a formula for g in terms of x (g is the x-pullback of the space time metric $g = -x^*G$).

(c) The symmetries of the action $S[g,x]$ are the same as for the length action $S[x]$.

Proof. (a) At the criticality equation for the inverse metric $g^{\tau\tau} = \frac{m^2}{-\dot{x}^2}$,

$$S[x,g]\big|_{g^{\tau\tau}=\frac{m^2}{-\dot{x}^2}} = -\frac{1}{2}\int_I d\tau\;(g^{\tau\tau})^{\frac{1}{2}}\dot{x}^2 - (g^{\tau\tau})^{-\frac{1}{2}}m^2\;\Big|_{g^{\tau\tau}=\frac{m^2}{-\dot{x}^2}}$$

$$= -\frac{1}{2}\int_I d\tau\;\frac{m}{\sqrt{-\dot{x}^2}}\dot{x}^2 - \frac{\sqrt{-\dot{x}^2}}{m}m^2 = -\frac{1}{2}\int_I d\tau\; -2m\sqrt{-\dot{x}^2} = \int_I d\tau\; m\sqrt{-\dot{x}^2}.$$

(b) The criticality for g is a formula for g in terms of x $g^{\tau\tau} = \frac{m^2}{-\dot{x}^2}$, which says that $g_{\tau\tau} = \frac{1}{m^2}x^*(-G)$. We can use it rewrite the criticality equation in x $\frac{\partial}{\partial\tau}[(g^{\tau\tau})^{\frac{1}{2}}\dot{x}] = 0$, as $0 = \frac{\partial}{\partial\tau}[\frac{m}{\sqrt{-\dot{x}^2}}\dot{x}]$, but this is exactly the EL-equation for the length action.

3.3.4. *Relativistic kinetic action for* **massless** *particles.* Now we consider the case when $m = 0$. Then the action is

$$S[x,g] \stackrel{\text{def}}{=} -\frac{1}{2}\int_I dl \quad (\dot{x})^2_{G,g}.$$

The criticality equation with respect to the path x: $\frac{\delta S}{\delta x} = 0$, is, as above, the requirement that the τ-velocity is a multiple of some constant vector c

$$\dot{x} = g_{\tau\tau}{}^{\frac{1}{2}}\cdot c.$$

The relation to the length action in the massive case was that the kinetic action reduces to the length action at the criticality in g^{-1}. However, in the massless case there should

be no relation since there is no length action for a massless particle. On the calculational level this is seen as the absence of solutions of the criticality equation with respect to the inverse metric – the equation is $g^{\tau\tau} = 0$.

3.4. **The philosophy of adding an auxiliary field to the action.** We are interested in

adding an auxiliary field y to action $S[x]$ while keeping the physics "equivalent".

Roughly, the new field will be eliminated either by the new criticality equation (when the action depends on it), or by the new symmetries (when it does not).

3.4.1. *Adding an auxiliary field y to action $S[x]$.* By this we will mean passing to a new action $\mathcal{S}[x, y]$ such that:

(1) $\mathcal{S}$ reduces to S at the criticality for y. (i.e., criticality gives $y = \phi(x)$ and $\mathcal{S}[x, \phi(x)] = S[x]$), and
(2) Actions are classically equivalent, i.e., solutions of $d_x S = 0$ and $d_x \mathcal{S} = 0 = d_y \mathcal{S}$ are the same.
(3) (???) The symmetries for $\mathcal{S}[x, y]$ are the same as for $S[x]$.

3.4.2. *Adding a "physically meaningless" field.* This is a simpler version. The new action $\mathcal{S}[x, y]$ simply does not depend on the new field y:

$$\mathcal{S}[x, y] = S[x].$$

So, y is really just a device for calculating the action $S[x]$ of x.

Part II. Quantum Mechanics

0. Intro

[7] We start with basic principles of quantum mechanics:

The probability of an event refines to a complex number called the amplitude. Amplitudes combine (by addition) in a much simpler way then the probabilities.

Our justification is based on (idealized) experiments with a finite number of possible evolutions of the system.

0.0.3. *Lagrangian approach.* When one applies these principles to systems with continuously many possible evolutions, the formula for the amplitude of a certain event transforms from a finite sum to an integral of amplitudes over all possible evolutions (=paths), i.e., a "path integral". The key is now to know the formula for the amplitude $Z[x]$ of a possible evolution, i.e., a path x. It turns out that this is a unit complex number $Z[x] = e^{i\hbar S[x]}$ with the phase equal to the action $S[x]$ of the path (up to the Planck constant $\hbar$). So this ties with the classical Lagrangian approach.

0.0.4. *Hamiltonian approach.* The fundamental experimental fact here is that the quantum configuration space – the space of states of a quantum system – has a structure of a Hilbert space $\mathcal{H}$. It usually has some functional interpretation such as $\mathcal{H} = L^2(\mathcal{C})$.

Moreover, the observables (in the sense of real functions on the momentum configuration space $T^*\mathcal{C}$), now become hermitian linear operators on $\mathcal{H}$. The evaluation of an observable f on a state v, i.e., the measurement $f(v)$ of f at v, now becomes the eigenvalue of the operator $\widehat{f}$ on the vector v. (For a more precise statement see the Dirac axioms of Q-Mechanics in 0.1 bellow.)

In particular, the Hamiltonian function H on $T^*\mathcal{C}$ now becomes the Hamiltonian operator $\widehat{H}$. Hilbert space comes with a hermitian operator

The physical law governing the system is now formulated as:

Time evolution of the system is generated by the Hamiltonian operator $\widehat{H}$.

The meaning is that the evolution of the system in time t is obtained by the unitary operator $U(t) = e^{\frac{2\pi i}{h}\widehat{H}t}$ acting on the configuration space $\mathcal{H}$. So, $\widehat{H}$ is the infinitesimal generator of time evolution.

0.0.5. *Functional realizations of $\mathcal{H}$.* In the example of *electron on a line* or in a vector space V, we will see that Hilbert space $\mathcal{H}$ can be realized as $L^2(V)$. Physicists use two traditional modes of computations (i.e., vies on the Hilbert space $\mathcal{S}$.

[7] There are two introductory sections for this part, and I need to fuse them!

- Use the subspace $\mathcal{T}$ of *nice states* (i.e., the *test functions*), and its continuous dual $\mathcal{D} = \mathcal{T}'$ of $\mathcal{T}$, which is the space of *idealized*, i.e., *distributional* states.
- Use algebraic structure on a dense countable-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ (or of $\mathcal{T}$). (This we will do for the *harmonic oscillator*, i.e., when the potential is a quadratic function of position $V = -x^2$.)

0.0.6. *Historical development: Hamiltonian to Lagrangian.* We will follow the historical development: we develop the operator approach to quantum mechanics, and then use it to derive the path amplitude formula.

Mathematicians often neglect the Hilbert space $\mathcal{H}$, i.e., they do not view its appearance as the fundamental event. Rather, they are impressed by the fact that the quantization of functions on $T^*\mathcal{C}$ to operators, happens in an organized way: the commutative algebra $\mathcal{O}(T^*\mathcal{C})$ of functions ("observables") on $T^*\mathcal{C}$, deforms to a non-commutative algebra $D_{\mathcal{C}}(\hbar)$. Then they find $\mathcal{H}$ as a natural representation of the algebra $D_{\mathcal{C}}(\hbar)$, hence they recover the operator interpretation of quantized functions. (However we have most difficulty with unitarity?)

The mathematical reason that the algebra $\mathcal{O}(T^*\mathcal{C})$ deforms is that $T^*\mathcal{C}$ has a canonical Poisson structure. So, we start with the mathematical framework for the Hamiltonian approach: the *deformation quantization of Poisson algebras* . For instance we find that $D_{\mathcal{C}}(\hbar)$ is the algebra of differential operators on $\mathcal{C}$, so $\mathcal{H}$ will be its natural representation $\mathcal{O}(\mathcal{C})$ (in various incarnations such as $L^2(\mathcal{C})$).

We will calculate the amplitudes in the operator approach and find that the result is a path integral. This traces the historical discovery of the precise formula for the integrand in the path integral. (However, after making this observation, Feynman gave an independent direct heuristic justification of the path integral amplitude formula.)

0.1. **Dirac axioms.**

(1) Deterministic axioms
   (a) To a physical system there corresponds a Hilbert space $\mathcal{H}$ (the configuration space of the system), and a Hermitian operator H on $\mathcal{H}$ (the Hamiltonian).
   (b) To every state ψ of the system there corresponds a "wave function" $\psi \in \mathcal{H}$, defined up to $\mathbb{C}^*$.[8]
   (c) If at time 0 the system is in the state ψ, then at a time t it is in the state $e^{\frac{2\pi i}{\hbar}tH}\psi$.

(2) Stochastic axioms related to measurements[9]
   (a) To any measurement process there corresponds a hermitian operator Q (an "observable"). If $\mathcal{H}$ has a basis v_i with Q-eigenvalues λ_i, then the only possible values of the measurement are the eigenvalues λ_i.
   (b) If the system is in a state $\psi = \sum a_i v_i$ (and we normalize v_i's), the probability that the measurement will give λ is $\frac{\sum_{\lambda_i=\lambda} |a_i|^2}{\sum_i |a_i|^2} \frac{\sum_{\lambda_i=\lambda} |a_i|^2}{|\psi|^2}$.
   (c) If the measurement produces value λ then the measurement process has moved the system from state ψ to the state v_i.

[8]The terminology is based on the functional realization of $\mathcal{H}$.

[9]Here, as in the original Dirac formulation, one assumes that the spectra are discrete and there are no multiplicities. A more precise version:

(a) To any measurement process there corresponds a hermitian operator Q (an "observable"). So, $Q = \int_{\mathrm{Spec}(Q)} \lambda \, dp^Q(\lambda)$ where p^Q is the associated projector valued measure. The only possible values of the measurement are the eigenvalues $\lambda \in \mathrm{Spec}(Q)$.

(b) If the system is in a state $\psi = \int \psi_\lambda \, d\lambda$ (with "ψ_λ in the eigenspace λ"), the probability that the measurement will give a value in the interval $[\lambda_1, \lambda_2]$ is

$$\frac{\int_{\lambda_1}^{\lambda_2} d\lambda \, |\psi_\lambda|^2}{|\psi|^2} = \frac{|p^Q_{[\lambda_1,\lambda_2]} \cdot \psi|^2}{|\psi|^2}.$$

(c) If the measurement produces value λ then the measurement process has moved the system from state ψ to the state $p^Q_\lambda \psi$.

1. Principles of Quantum Mechanics

Text. R.P. Feynman, *Quantum Mechanics,* volume **4** of *Feynman's lectures on Physics.*

1.1. **Amplitudes.** We have no natural experience (intuition) about what happens on the small scale.[10] By analyzing certain particle experiments one arrives at the following picture of quantum systems, i.e., systems on a small scale. This is common to the Lagrangian and Hamiltonian approach.

1.1.1. *Complex numbers and amplitudes.* The complex numbers appear in analogy with the water waves – these are known to require complex numbers in other to describe interactions. In quantum mechanics they appear as "probability amplitudes" which are refinements of the probability of an event. This gives There is a "pairing" or "inner product operation" on states, given by the

probability amplitude $\langle j|i\rangle \in \mathbb{C}$, for a transition from one state i into another state j.

Amplitude is sesquilinear: linear in i and anti-linear in j.

1.2. **Hilbert space setting (the Hamiltonian view).**

1.2.1. *The linear algebra of a Hilbert space.* The notion of the "state of the system" is the analogue for quantum systems of a point of the configuration space – it is a complete description of the system at a certain time. A fundamental observation:

the set $\mathcal{H}$ of all states of the system forms a Hilbert space,

is obtained in the following way:

(1) A choice of a "filtering apparatus" S (an experimental machinery), gives the family B_S of "*S-pure*" states. These pure states are orthonormal for the probability amplitude pairing. Moreover, we can assume that these pure states can not be filtered into simpler states (for this, if need be, we replace a simple filtering apparatus S by a succession $S = (S_1, ..., S_p)$ of of such filtering apparata.)
(2) The additive structure of the set $\mathcal{H}$ of all states of the system comes from the further experimental fact that for any state ϕ, amplitudes $\langle \phi, i\rangle$, $i \in B_S$, behave as coefficients of a vector ϕ in the basis B_S.

1.2.2. *Appearance of linear operators.* Starting from the notation $\langle x|s\rangle$ for the probability amplitude one uses symbol $|s\rangle$ to denote state s, or the corresponding vector. An operation A on states can be described in terms of the amplitudes $\langle j|A|i\rangle$ for the probability that operation A on a particle in the state i will give a particle in the state j. Again, we check experimentally that the amplitudes $\langle j|A|i\rangle$ are sesquilinear in i and j (linear in i

[10] It may be difficult to admit.

and anti-linear in j), so we can think of such operation A as a linear operator $|s\rangle \mapsto A|s\rangle$ on the space of states. This interpretation of

amplitudes as matrix coefficients

is the heart of the *Hamiltonian approach.*

1.2.3. *Time evolution by the Hamiltonian operator (Schroedinger's equation).* Next experimental act is the time evolution $\psi(t)$ of a state ψ has an infinitesimal generator, a linear operator $\widehat{H}$ which we call the Hamiltonian operator:

$$\frac{d}{dt}\psi(t) \;=\; \frac{2\pi i}{\hbar}\widehat{H}\;\psi.$$

This we call the *abstract Schroedinger's equation.* To apply this formalism, one needs to say what $\widehat{H}$ is in a given situation. This gives a particular Schroedinger equation, which is a differential equation that describes the evolution of the quantum system.

Historically, this breakthrough came in 1926 when Schroedinger described by such equation the motion of electrons on atomic scale. The above conceptual framework came later, and it become the foundation only with Feynman's book.

1.3. **Feynman integrals (the Lagrangian view).** Feynman's point of view on quantum mechanics is that the amplitude for an event is total of amplitudes over all possible evolutions $\boldsymbol{x}$ of the system that result in this event. In the Minkowski setting (physically reasonable), this is an oscillating integral

$$\int_{\boldsymbol{x}} d\boldsymbol{x} \quad e^{\frac{i}{\hbar}S[\boldsymbol{x}]}\ .$$

The amplitude for the evolution $\boldsymbol{x}$ is a unit complex number $e^{\frac{i}{\hbar}S[\boldsymbol{x}]}$ where S is the action.

The relation to the classical mechanics comes from the *Principle of Stationary Phase*, which says that the leading contributions to an oscillating integral come from points where phase does not vary. So the classical solutions – evolutions $\boldsymbol{x}$ such that $d_{\boldsymbol{x}}S = 0$ give the main contribution to the amplitude. In other words, it is most likely that the system will evolve according to a classical solution, but other evolutions also contribute.

We will follow the historic appearance of Feynman integrals:

1.3.1. *Operator approach leads to Feynman integrals.* When we try to account for the amplitude of a certain transition from time t_i to time t_f, as an accumulation of contributions over small time subintervals $dt = t_{k-1}, t_k]$, we find that the basic principles on how the amplitudes interact give contributions from all possible positions x during dt, and all possible momenta p during dt. So, the amplitude is an integral over time t and all possible choices $x(t), p(t)$ of the position and momentum at time t. So we really get the integral over all curves (x, p) in $T^*\mathcal{C}$.

The integrand happens to be exactly $e^{\frac{i}{\hbar}\mathcal{S}[x,p]}$ for some action $\mathcal{S}$ on paths (x,p) in $T^*\mathcal{C}$. This comes from the formula for the Hamiltonian operator as a quantization of the Hamiltonian function, which is of course related to the Lagrangian.

2. Amplitudes

2.1. **Shooting electrons through holes leads to:** *probability, complex numbers, uncertainty principle and the dual nature of particles*.

2.1.1. *Experiment.* In an idealized experiment, suppose that a source shoots something towards the wall with holes 1 and 2. A bit behind the 1st wall is the second parallel wall, with a detector that counts the hits at a position x (a coordinate on the second wall). Consider the probability distributions

- $p(x)$ for the hit to be at x, and
- $p_i(x)$ (for $i = 1, 2$), for the hit to be at x if only the hole i is open.

2.1.2. *Shooting bullets.* Assume we shoot some kind of bullets which can be deflected in an arbitrary direction when passing thru the hole, but they do not break.

So, bullets they arrive in quanta – a bullet at a time – (no halves etc) and clearly $p = p_1 + p_2$ which we describe by "no interference". (The contributions from each possibility add up, so there is no interaction between the two kinds of events.)

2.1.3. *Shooting water waves.* For water waves, we measure not the probability - the wave will certainly arrive – but the intensity $I(x)$ at x, which is defined as the mean square of the height (energy) of the wave. Here, the arrival is a continuous quantity. Notice that now two waves are going to interfere at x.

The height of a single wave at x is of the form

$$\mathcal{H}(x,t) = C(x)\cos(\omega t - p) = A(x)\cos(\omega t) + B(x)\sin(\omega t)$$

(harmonic oscillator!), which one can rewrite as

$$\mathcal{H}(x,t) = Re[(A(x) - iB(x))e^{i\omega t}] = Re[h(x)e^{i\omega t}], \quad \text{where} \quad h(x) \stackrel{\text{def}}{=} A(x) - iB(x).$$

We will call the complex number $h(x)$ the "amplitude of intensity" at x. Then the intensity at x is going to be

$$I(x) \stackrel{\text{def}}{=} \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} \mathcal{H}(x,t)^2\,dt = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} A^2\cos^2(\omega t) + B^2\sin^2(\omega t) - AB\sin(2\omega t)\,dt$$

$$= \frac{1}{2\pi/\omega}(A^2+B^2)\int_0^{2\pi/\omega}\cos^2(\omega t)\,dt = \frac{1}{2\pi/\omega}(A^2++B^2)\int_0^{2\pi/\omega}\frac{1}{2}2\pi/\omega = \frac{1}{2}\,|h(x)|^2.$$

The way the intensities add up when two waves (with the same frequency ω) interfere, is complicated, the difference δ in the phases gives $I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$ (the "cosine rule in a triangle"). However, the formula for the addition of complex amplitudes is simple: $h(x) = h_1(h) + h_2(x)$.

2.1.4. *Shooting electrons.* If one shoots electrons, they will arrive in quanta (only the whole electrons), but the probabilities $p_i(x)$ at the position x have the same interference pattern as the intensities of water waves.

(For the comparison of the three experiments, notice that if one shoots bullets, water waves or electrons; the graphs of p_i's will always be the same (a random distribution), while the graphs of $p(x)$ will be the same only for water waves and electrons.

This suggests that electrons have dual nature in the sense that besides their standard behavior as particles ("bullet-like"), they also sometimes behave as waves. Moreover, we see that one will need complex numbers to describe particles. Once this "correct" point of view is taken – that the probability is originally a complex number – we return to the "no interference" situation (now for amplitudes rather then for probabilities themselves).

2.1.5. *Uncertainty principle.* This is the principle that our observation necessarily *affects* the experiment! We see that this has to be happening because if one can actually observe thru which of the holes the electron is passing, then clearly $p = p_1 + p_2$ as for the bullets.

2.2. **The amplitude principles.** These are basic principles of Quantum Mechanics:

- (0) (Amplitude axiom) The probability p of an event can be refined to a complex number ϕ called probability amplitude, they are related by
$$p = |\phi|^2 \quad \text{and} \quad \phi = \sqrt{p}e^{i\delta},$$
So, the amplitude ϕ consists of the probability p and the phase δ, and always $|\phi| \leq 1$.
- (1) (Combining amplitudes) If an event can happen in two different ways then its amplitude is
$$\phi = \phi_1 + \phi_2.$$
So, probability $p = \sqrt{(p_1)^2 + (p_2)^2 + 2p_1p_2\cos(\delta_1 - \delta_2)}$depends on the relative phase.
- (2) (Concatenation) The amplitude for the particle to arrive from i to k via j is the product of the amplitude for the particle to go from i to j and the amplitude for the particle to go from j to k.
- (3) (Reversal) The amplitude for the particle to arrive from i to k is the complex conjugate of the amplitude for the particle to arrive from k to i.

The above axioms encode a lot, so we can use them as a foundation and derive ideas (like the Uncertainty axiom), without invoking particular experiment.

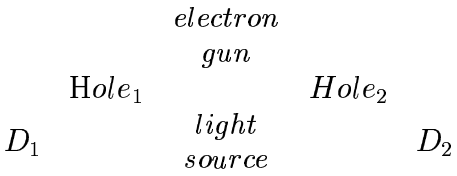
2.2.1. *Complex numbers.* We see now how elementary experimental facts construct all aspects of complex numbers from nature. While (1) uses only a 2-dimensional vector space $\mathbb{R}^2$ with an inner product, (2) uses multiplication and (3) uses the real structure on $\mathbb{C}$, so we have recovered the whole structure of complex numbers from physics!

2.3. **Uncertainty principle: a quantitative version.** We will now try to observe through which hole did the electron go through. For the observation we use the light, i.e., a photon, of a certain wavelength. We will see that two features of the situation

- The result of the measurement, i.e., what we it tells us has happened,
- How efficient the apparatus is;

are both dependent on the choice of wavelength. This degree of freedom gives us quantitative view on the relation between these two features. The result will be that in order to make the apparatus more efficient (so that it is likely to report a definite result), we have to affect more the result of the measurement.

2.3.1. *The experimental setting.* Behind the 1st wall, let us add a source of light (emitting photons),l in the middle position, and two light detectors D_1 and D_2, near holes H_1 and H_2. So one has



For $i, j \in \{1, 2\}$, we will denote the amplitude of basic events by

- $\langle i|g\rangle = \langle H_i|g\rangle$ is the amplitude for electron to shot out of the electron gun to pass through the hole H_i,
- $\langle x|i\rangle = \langle x|H_i\rangle$ is the amplitude for electron that went through the hole H_i to go to the position x,
- a_{ij} is the probability amplitude for an electron which has passed through the hole H_i, to scatter a photon to the detector D_j. By symmetry,

$$a_{11} = a_{22} \stackrel{\text{def}}{=} \alpha \quad \text{and} \quad a_{21} = a_{12} \stackrel{\text{def}}{=} \beta.$$

The influence of the wavelength:

(1) efficient apparatus case: if the *wavelength is small* , then approximately

$$\beta = 0.$$

(2) inefficient apparatus case: if the *wavelength is large*, then approximately

$$|\alpha|^2 = |\beta|^2.$$

We will also assume that the phases of α and β agree.

2.3.2. *Calculation of some amplitudes.* First, the amplitude $\phi_i(x)$ and three probability $p_1(x)$ of the electron to go from the gun through the hole H_i *and* go to the position x (the second hole may as well be closed), is

$$\phi_i(x) \;=\; \langle x|H_i\rangle \cdot \langle H_i|g\rangle \quad \text{and} \quad \psi_i(x) \;=\; |\phi_1(x)|^2.$$

Next, our measuring device, will *tell us* that the electron went through the hole H_j and to the position x if the photon shows up at the detector D_j and the electron at the position x. This happens in two cases $(i = 1, 2)$, when

electron goes from g to x via the hole H_i and at H_i it scatters a photon to detector D_j.

The amplitude for each of these two events is

$$\langle x|i\rangle \cdot a_{ij} \cdot \langle i|g\rangle = a_{ij} \cdot \phi_i(x).$$

So, the amplitude for the detectors to tell us that the electron went to go through the hole $j = 1$ to the position x is

$$\sum_i a_{ij} \cdot \langle x|i\rangle \cdot \langle i|g\rangle \;=\; \sum_i a_{i1} \cdot \phi_i(x) \;=\; a_{11}\phi_1(x) + a_{21} \cdot \phi_2(x) \;=\; \alpha \cdot \phi_1(x) + \beta \cdot \phi_2(x).$$

The corresponding probability of the measurement to tell us that the electron went thru the hole $j = 1$ to the position x is therefore,

$$\boldsymbol{p} \;=\; |\alpha \cdot \phi_1 + \beta \cdot \phi_2|^2.$$

The efficient case. Here *wavelength is small* , β is close to 0 and

$$\boldsymbol{p} \cong \; |\alpha \cdot \phi_1|^2 \;=\; |\alpha|^2 \cdot p_1.$$

The efficient case. Here *wavelength is large*, $|\alpha|^2 \cong |\beta|^2$, and if we also assume that the phases of α and β agree then

$$\boldsymbol{p} \;\cong\; |\alpha|^2 \cdot |\phi_1 + \phi_2|^2.$$

2.3.3. *Conclusion.* Let us count the number of hits at x in the presence of the measurement apparatus, and see whether it depends on the efficiency of the apparatus.

If we divide by the light effect $|\alpha|^2$, in the *efficient* case we get the correct result $p_1 = \phi_1(x)|^2$, so the total number of hits at x will be the sum

$$|\alpha|^2 \cdot (p_1(x) + p_2(x)).$$

In the *inefficient* case we get the interference formula. and the total number of hits at x will be

$$|\alpha|^2 \cdot 2|\phi_1 + \phi_2|^2,$$

showing the interference pattern. So,

The nature of our observation apparatus influences the events,
a more efficient apparatus causes a more classical behavior.

3. **Operators: the Hamiltonian approach**

The basic idea here is that the configuration space for a quantum system is a Hilbert space. This is deduced from experiments in 3.2. We find that the operations on states are given by linear operators, and in particular that the evolution of the system is given by one operator that we call the Hamiltonian operator (3.3).

The most relevant mathematical theory is the deformation quantization which we visit in 3.1. To mathematicians, the most reasonable aspect is the fact that the quantization of functions on $T^*\mathcal{C}$ to operators, happens in an organized way: the commutative algebra $\mathcal{O}(T^*\mathcal{C})$ of functions ("observables") on $T^*\mathcal{C}$, deforms to a non-commutative algebra $D_{\mathcal{C}}(\hbar)$. The mathematical reason that the algebra $\mathcal{O}(T^*\mathcal{C})$ deforms is that any cotangent bundle has a canonical geometric structure: the Poisson structure. Once one finds $D_{\mathcal{C}}(\hbar)$, they recover the Hilbert space $\mathcal{H}$ as its natural representation. This usually gives a functional interpretation of $\mathcal{H}$ and therefore leads to a Functional Analysis.

3.1. **Deformation quantization: a mathematical notion of quantization.**

3.1.1. *Example: functions on the cotangent bundle deform to differential operators.* For each $\hbar \in \mathbb{C}$ consider the algebra $\mathcal{A}_\hbar$ generated by p and q and the relation $pq - qp = \hbar \cdot 1$. In each of these, monomials $q^i p^j$ form a basis, so we can think of them as a family of associative algebras with 1 on one vector space A. We will think of $\mathcal{A}_1$ as the ring of differential operators $D_{\mathbb{A}^1}$ on the line $\mathbb{A}^1$ with coordinate q and $p = \partial_q$. At the other extreme, $A_0 = \mathbb{C}[p,q]$ is the algebra of polynomial functions on $\mathbb{A}^2$ and we will think of this $\mathbb{A}^2$ as $T^*\mathbb{A}^1$ with with q a coordinate on the space $\mathbb{A}^1$ and p a coordinate in the vertical direction.

The same works for $\mathbb{A}^n$ and it is invariant under a change of coordinates, so it actually works on any manifold M.

3.1.2. *Deformation of a commutative algebra gives a Poisson structure.* Consider a vector bundle $\mathcal{A}$ over a neighborhood U of 0 in $\mathbb{C}$, and a family of associative algebra structures $\underset{\hbar}{\cdot}$ on vector spaces $\mathcal{A}_\hbar$, $\hbar \in U$, such that the multiplication $a \underset{\hbar}{\cdot} b$ is an analytic function on $\mathcal{A}$. We say that this is a deformation of the algebra $A = \mathcal{A}_0$ on U.

Since one can trivialize the vector bundle $\mathcal{A}$ near $\hbar = 0$, this is equivalent to the following simpler definition. A deformation of the algebra A over U is a family of associative algebra structures $\underset{\hbar}{\cdot}$ on the vector space A, such that (0) $(A_0, \underset{0}{\cdot}) \cong A$ and (i) the multiplication $a \underset{\hbar}{\cdot} b$, $a, b \in A$, $\hbar \in U$, is an analytic function of $\hbar$, i.e., near 0

$$a \underset{\hbar}{\cdot} b = \sum_{n=0}^{\infty} \hbar^n \, a \overset{n}{\cdot} b, \quad a, b \in A.$$

3.1.3. *The ordering problem.* The second point of view is more convenient for calculations. For instance now we can say that $f \in A = \mathcal{A}_0$ quantizes (deforms) to f in the algebra $A = \mathcal{A}_\hbar$. However. this point of view is non-canonical since in practice the vector bundle $\mathcal{A}$ does not have a natural trivialization.

For instance in our main example, the identification of vector spaces $\mathcal{O}(T^*\mathbb{A}^n) \cong D_{\mathbb{A}^n}$ is canonical only on certain generators q_i, p_j and x_i, ∂_j. Since there is no canonical identification for the whole vector spaces, various trivializations leads to various possible quantization $\widehat{f} \in AA_1$ of a given $f \in \mathcal{A}_0$.

We call this the ordering problem since in this example, a specification of the ordering of q_i, p_j's gives a basis of all algebras $\mathcal{A}_s$, hence a trivialization of the vector bundle $\mathcal{A}$.

Our main interest is the correct quantization $\widehat{H}$ of the Hamiltonian function H on $T^*\mathcal{C}$. Notice that there is no problem if there are no products of p and q: $H(q,p) = H_{potential}(q) + H_{kinetic}(p)$.

Lemma. Any deformation of a commutative algebra A makes A into a Poisson algebra by

$$\{a,b\} \stackrel{\text{def}}{=} \lim_{\hbar\to 0} \frac{a \underset{\hbar}{\cdot} b - b \underset{\hbar}{\cdot} a}{\hbar},$$

3.1.4. *(Deformation) quantization of Poisson algebras.* We will reverse the point of view and suppose that we start with a Poisson algebra $(A, \{,\})$ (a commutative algebra with a Poisson structure). A (deformation) quantization of the Poisson algebra $(A, \{,\})$ will be any deformation $\mathcal{A}$ of $A \cong \mathcal{A}_0$, such that the Poisson structure on $A = \mathcal{A}_0$ induced by the deformation is $\{,\}$. One also often says imprecisely that a specific algebra $\mathcal{A}_1$ in the family $\mathcal{A}_\hbar$ is a quantization of $(A, \{,\})$.

3.1.5. *Differential operators are a quantization of functions on the cotangent bundle.* On any cotangent bundle, there is a canonical Poisson structure (I.2.6.2). We used it for the "canonical formalism" formulation of classical mechanics. The above deformation of functions on the cotangent bundle (3.1.1), is a quantization of the canonical Poisson structure on the cotangent bundle, since the Poisson structure on $\mathcal{O}(T^*\mathcal{A}^1)$ obtained from the deformation is the canonical Poisson structure:

$$\{p,q\} \stackrel{\text{def}}{=} \lim_{\hbar\to 0} \frac{p \underset{\hbar}{\cdot} q - q \underset{\hbar}{\cdot} p}{\hbar} = \lim_{\hbar\to 0} \frac{\hbar \cdot 1}{\hbar} = 1.$$

3.1.6. *Comparison with a quantization of a system in physics.* In physics, the basic example of quantization is that of the harmonic oscillator. In the Hamiltonian approach it involves the deformation of the algebra of functions on the phase space $T^*\mathcal{C}$ (a Poisson space) to the algebra of differential operators $D_\mathcal{C}$ on the configuration space. However, more is needed:

- A Hilbert space $\mathcal{H}$ with an action of the quantization algebra $D_{\mathcal{C}}$. (In this example we can use $\mathcal{H} = L^2(\mathcal{C})$.)
- An interpretation of $\mathcal{H}$ as the space of (quantum) states – the quantum analogue of the configuration space $\mathcal{C}$.
- A particular "quantization" of the Hamiltonian function H on $T^*\mathcal{C}$ to an "operator" $\widehat{H} \in D_{\mathbb{A}^n}$ acting on $\mathcal{H}$. (It gives the time evolution of the quantum system $v \mapsto e^{it\widehat{H}}v$.)
- Particular quantizations $\widehat{f} \in D_{\mathbb{A}^n}$ of interesting observables $f \in \mathcal{O}(T^*\mathcal{C})$.

3.1.7. *Unitarity problem.* A number of questions arise, such as *unitarity*, i.e., we need a

(1) positive definite inner product on $\mathcal{H}$, and
(2) it should be compatible with the quantization of observables in the sense that for a real valued function f, $\widehat{f}$ should be a Hermitian operator.

This is needed to ensure both that the eigenvalues of $\widehat{f}$ are real and plentiful, and this is essential since the measurement of f on a state v of the system now makes sense only if $v \in \mathcal{H}$ is an eigenvector of $\widehat{f}$. Then one says that the value of the observable f on the quantum state v is the corresponding eigenvalue.

The unitarity needs force the following quantization convention on generators[11]

$$[\widehat{p}, \widehat{q}] = \frac{\hbar}{i}\widehat{\{p, q\}}.$$

So, since the Poisson structure on $T^*\mathbb{A}^1$ is given by $\{p, q\} = 1$, we get $[\widehat{p}, \widehat{q}] = \frac{\hbar}{i}$.

3.2. **Stern-Gerlach filtering apparata leads to** *Hilbert space structure on the space of states***.** Consider the configuration space $\mathcal{H}$ of some quantum system. We will find that the experiments endow it with a Hilbert space structure.

3.2.1. *Dirac notation $\langle j|i\rangle$ for amplitudes.* Dirac's notation for the probability amplitude of the particle in state j to arrive into the state i is $\langle j|i\rangle$. [12] The amplitude axioms (2) and (3) are now written as

$$\langle k|i\rangle_{\text{via } j} = \langle k|j\rangle \cdot \langle j|i\rangle \qquad \text{and} \qquad \langle k|i\rangle = \overline{\langle i|k\rangle}.$$

[11] This means that $[\frac{i}{\hbar}\widehat{p}, \frac{i}{\hbar}\widehat{q}] = \widehat{\frac{i}{\hbar}\{p, q\}}$, i.e., the natural quantization of p is really $\frac{i}{\hbar}\widehat{p}$. Say, the Hamiltonian function H is real valued but the interesting operator is the infinitesimal generator of time, so it is the derivative of a unitary representation, and therefore it is anti-hermitian. It is denoted $\frac{i}{\hbar}\widehat{H}$.

[12] Here we do not yet worry about the time it takes, the only question is whether certain procedure will end up with the state x. However, it is not clear what this amplitude really is, i.e., does it depend on a particular procedure or what?

3.2.2. *Pure states form a bases of the Hilbert space.* We will consider the example of the *spin* of a particle and we will use the (idealized) Stern-Gerlach apparatus to analyze the notion of a *spin*. Since we are only interested in the spin, the state of a particle will mean its spin (and we design the experiment cleverly so that no other feature of the situation interferes. The final conclusion will be that the spin, i.e., any possible state of the spin of a particular particle, lies in a certain finite-dimensional vector space (an irreducible representation of $SL_2(\mathbb{C})$ or $SU(2)$).

A Stern-Gerlach apparatus consists of a (non-homogeneous) magnet thru which passes a beam of particular particles. If the apparatus splits the beam into n beams, we say that this kind of particles has spin $\frac{n-1}{2}$. Say, if there is no splitting ($n = 1$), then the spin is 0. The spin is $\frac{1}{2}$ if there are two beams.

In the case of a spin one particle we get three beams which are traditionally described as

- the up beam called $+$, its angular momentum is $\hbar$,
- the down beam called $-$, with angular momentum $-\hbar$, and
- the undisturbed part of the beam called 0, with angular momentum is 0.

For a particular apparatus S of such type, we can say that a particle is in the "pure state i for S" (we denote this state i_S), if it "naturally belongs to the i^{th} beam". This is a *property* of the particle: if we apply again the apparatus S to the i^{th} beam, this time it will only produce the i^{th} beam.

The spectacular experimental fact is that each particle will go into one of these three beams, the apparatus will find it in some pure S-state i. So, for any state ϕ, the experiment will push it into one of the beams. We can measure the probability amplitude for a given state ϕ to go to the i^{th} beam, i.e., the amplitude $\langle i|\phi\rangle$ for the apparatus to find ψ to be in the S-state i.

It turns out that these amplitudes behave as coefficients in the decomposition of a vector ϕ into a linear combination of pure S-states. To see that this is meaningful, one compares the coefficients for various choices of the apparatus. Let us follow the apparatus S with another one of the same nature but a different build (direction), call it T. Then the i^{th} S-beam will split into T-beams and we encode this in the "transition matrix" of amplitudes, denote

$$(TS)_{ji} = \langle j_T|i_S\rangle$$

the amplitude for a particle in the i^{th} S-beam to end up in the j^{th} T-beam. One now checks that these transition matrices have the expected algebraic properties – they behave as the change of bases matrices for a vector space[13].

[13]Some algebraic properties of matrices: (0) (SS) is the identity matrix 1 and in general, transition matrix satisfies (i) $(ST) = \overline{(TS)}$ (reversal) and (ii) $(ST)(TS) = 1$ **on the diagonal**, (since $\sum_j (ST)_{ij}\cdot(TS)_{ji} = \sum_j |(TS)_{ji}|^2$ is the probability that the particle goes from i_S to *some* T-state, i.e., 1).

3.2.3. *The conclusion.*

- The space of states is a vector space $\mathbb{H}$ over complex numbers.
- The probability amplitude is a positive definite inner product $\langle -|-\rangle$ on $\mathcal{H}$ (linear in the second and anti-linear in the first variable).
- Each Stern-Gerlach apparatus S chooses an orthonormal bases $i_S,\ i \in \mathbb{B}_S$, of the space of states $\mathcal{H}$.
- Any physical operation A on states is a linear operator on the space of states $\mathcal{H}$.

For the last point, starting from the notation $\langle x|s\rangle$ for the probability amplitude one uses symbol $|s\rangle$ to denote state s, or the corresponding vector. An operation A can be described in terms of the amplitudes $\langle j|A|i\rangle$ for the probability that operation A applied to a particle in the state i, will produce a particle in the state j. Once we check the sesquilinearity in i and j, we know that these amplitudes are matrix coefficients of a linear operator.

In any situation more complicated then the above study of spins, we may have to use several apparata in succession to filter the atomic system into the *pure states*, that can not be decomposed further. Then the algebra is the same.

3.2.4. *Notation.* Starting from the notation $\langle\phi|\psi\rangle$ for amplitudes, we often denote a state ψ in $\mathcal{H}$ also by $|\psi\rangle$. By $\langle\psi|$ we denote the the corresponding linear functional obtain by putting ψ in the first argument of the inner product.

In this notation, the action of a linear operator A on $\mathcal{H}$ and its hermitian adjoint A^* are denoted

$$A|\psi\rangle \stackrel{\text{def}}{=} |A\psi\rangle \quad \text{and} \quad \langle\psi|A \stackrel{\text{def}}{=} \langle A^*\psi| \stackrel{\text{def}}{=} \langle\psi|A-\rangle.$$

The meaning of the triple symbol is now clear

$$\langle\psi|A|\phi\rangle \stackrel{\text{def}}{=} \langle\psi|A\phi\rangle = \langle A^*\psi|\phi\rangle.$$

In particular, for a hermitian operator H and a unitary operator U one has

$$\langle\psi|H|\phi\rangle = \langle\psi|H\phi\rangle = \langle H\psi|\phi\rangle \quad \text{and} \quad \langle\psi|U|\phi\rangle = \langle\psi|U\phi\rangle = \langle U^{-1}\psi|\phi\rangle.$$

3.3. **Time evolution of a quantum system is given by the Hamiltonian operator $\widehat{H}$.** Here we switch the point of view from experiments in which the time did not have any influence (the total effect of a certain apparatus we designed) to an undisturbed (by us) evolution of a system. The effect of time on states is an operation that can be thought of in terms of linear transforms $U(t_2, t_1)$ on the space of states.

One can ask does this means that the states "really" evolve under the action of U if there is no disturbance (such as measurements), or that this is the "ideal" or most likely evolution of the system (while the actual evolution is not really predictable). I think that the only meaningful (measurable) quantity are the matrix coefficients $\langle\psi|U(t_2, t_1)|\phi\rangle$, i.e., the amplitudes for ϕ to evolve to ϕ from t_1 to t_2. So, U is really a shorthand for a bunch

of numbers called amplitudes, that (we imagine) describe the probabilities of possible evolutions.

3.3.1. *Lemma.* (a) Operators $U(t_2, t_1)$ are unitary.

(b) If the physical system is time independent then $U(t) = e^{itH}$ for a (densely defined) Hermitian operator $H = \frac{d}{dt}|_{t=0}\, U(t, 0)$.[14]

Proof. (a) is the claim that the time evolution preserves amplitudes $\langle U(t_2, t_1)\phi, U(t_2, t_1)\psi\rangle = \langle \phi, \psi\rangle$.

(b) Time independence: means that $U(t_2 + t, t_1 + t) = U(t_2, t_1)$. Then $U(t_2, t_1) = U(t_2 - t_1)$ for $U(t) \stackrel{\text{def}}{=} U(t, 0)$. The consistency condition for the time evolution $U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)$, now says that $U(t)$ is a "one-parameter unitary group", i.e., $U(t' + t'') = U(t')U(t'')$. So $U(t)$ is determined by values on a small interval near 0, more precisely, the functional analyses says that $U(t) = e^{it\widehat{H}}$ for a (densely defined) Hermitian operator $\widehat{H} = \frac{d}{dt}|_{t=0}\, U(t)$. It is called the Hamiltonian operator since it will turn out to be the quantization of the Hamiltonian function. $\widehat{H} = \frac{d}{dt}|_{t=0}\, U(t)$.

3.3.2. *Notation: time dependence of states.* $\mathcal{H}$ is the configuration space of all possible states of the system. We will relate it to time 0 by denoting

$$|\psi, t\rangle \stackrel{\text{def}}{=} \text{ the state which from time 0 to time } t \text{ evolves to } \psi\rangle$$

We will call it the "state ψ at time t". Since $|\psi\rangle = U(t)|\psi, t\rangle$, this is

$$|\psi, t\rangle = U(t)^{-1}|\psi\rangle.$$

In particular, we will be interested in the "pure position state x at time t", i.e., $|x, t\rangle|x_{\text{po}}, t\rangle = U(-t)|x_{\text{po}}\rangle$.

3.4. **Example: a single electron in a vector space.** Consider an electron in a vector space V. We postulate that

- the Hilbert space $\mathcal{H}$ has two "continuous basis" given by the pure position states $|v_{\text{po}}\rangle,\ v \in V$; and the pure momentum states $|p_{\text{po}}\rangle,\ p \in V^*$.
- The amplitudes between position and momentum states ($u, v \in V,\ \alpha, \beta \in V^*$), are

$$\langle v_{\text{po}}|u_{\text{po}}\rangle = \delta(u - v), \quad \langle \beta_{\text{mo}}|\alpha_{\text{mo}}\rangle = \delta(\alpha - \beta) \quad \text{and} \quad \langle \alpha_{\text{mo}}|x_{\text{po}}\rangle = e^{-\frac{i}{\hbar}\langle \alpha, v\rangle}.$$

This means that (i) $L^2(V) \cong \mathcal{H} \cong L^2(V^*)$, and that (ii) the transition between these two interpretations of $\mathcal{H}$ is given by the Fourier transform.

Let us first consider an electron on a line.

[14]It is called the Hamiltonian operator since it will turn out to be the quantization of the Hamiltonian function. However, here we only found a logical need for the existence of such infinitesimal generator of time evolution.

3.4.1. *Idealized position states and momentum states.* One starts with the (idealized) states $|x\rangle = |x_{\mathrm{po}}\rangle,\ x \in \mathbb{R}$ in which the *position* is absolutely known to be x (the index $_{\mathrm{po}}$ stands for position).

The states $|x_{\mathrm{po}}\rangle,\ x \in \mathbb{R}$ form a continuous analogue of the pure bases states in a quantum system with a finite dimensional space of states (3.2). We can think of these *position states* as a *continuous basis* $|x_{\mathrm{po}}\rangle,\ x \in \mathbb{R}$, of the space of states. The idealized, i.e., distributional, nature of the states $|x_{\mathrm{po}}\rangle,\ x \in \mathbb{R}$, occurs in the amplitudes

$$\langle y_{\mathrm{po}}|x_{\mathrm{po}}\rangle = \ \delta(x-y)$$

which make sense only as distributions. The precise meaning of this being a continuous basis is that for any state ψ we now have

$$|\psi\rangle \ = \ \int_{\mathbb{R}} \langle x_{\mathrm{po}}|\psi\rangle \cdot x_{\mathrm{po}}\ dx,$$

in the "weak" sense that this is true for amplitudes

$$\langle \phi|\psi\rangle \ = \ \int_{\mathbb{R}} \langle x_{\mathrm{po}}|\psi\rangle \cdot \langle \phi|x_{\mathrm{po}}\rangle\ dx,$$

There is another "basis" consisting of the similar (idealized) states $|p_{\mathrm{mo}}\rangle$ such that the momentum is known to be p ($p \in \mathbb{R}$).

3.4.2. *Transition amplitudes.* [15] The two continuous bases are related by the amplitudes

$$\langle p_{\mathrm{mo}}|x_{\mathrm{po}}\rangle = \ e^{-\frac{i}{\hbar}p\cdot x}.$$

3.4.3. *The functional interpretation of states as wave functions.* Since any state ψ is determined by the amplitudes $\langle \psi|x_{\mathrm{po}}\rangle,\ x \in \mathbb{R}$, we can think of the state ψ as a *function*

$$\psi(x) \stackrel{\mathrm{def}}{=} \langle x_{\mathrm{po}}|\psi\rangle \ \text{ on } \mathbb{R}.$$

For instance, $p_{\mathrm{mo}}(x) = \ \langle x_{\mathrm{po}}|p_{\mathrm{mo}}\rangle = \ \overline{\langle p_{\mathrm{mo}}|x_{\mathrm{po}}\rangle} = \ e^{\frac{i}{\hbar}p\cdot x}$.

Lemma. (a) The functional interpretation of states gives an identification of Hilbert spaces $\mathcal{H} \stackrel{\cong}{\to} L^2(V),\ \psi \mapsto \psi(x)$. In terms of the continuous bases $|x_{\mathrm{po}}\rangle,\ x \in V$, of the space of states, the amplitude is the L^2-inner product $\langle \phi|\psi\rangle = \ \int_V \ \phi(x)\overline{\psi(x)}\ dx$.

(b) The transition between writing a state in terms of pure position states $|x_{\mathrm{po}}\rangle$ and the pure momentum states $|p_{\mathrm{mo}}\rangle$ is the Fourier transform.

[15] What is the meaning of these? For instance why is not invariant under translations in V (so there is a distinguished position)? Certainly it should not be in invariant under V^*-translations, i.e., the change of momentum.

(c) With the functional interpretation of states, any "nice" operator A on states, can be written as an integral operator

$$(A\psi)(y) \int dx\ \psi(x)\cdot A(x,y) \quad \text{with the kernel} \quad A(x,y) \stackrel{\text{def}}{=} \langle \delta_x | A | \delta_y \rangle.$$

Proof. (a) $\langle \phi|\psi\rangle = \ \langle \int_V dx\ \langle x_{\text{po}}|\phi\rangle\cdot|x_{\text{po}}\rangle \big| \int_V dy\ \langle y_{\text{po}}|\phi\rangle\cdot|y_{\text{po}}\rangle$

$$= \int_{V^2} dx\ dy\ \langle \phi(x)\cdot|x_{\text{po}}\rangle \big| \psi(y)\cdot|y_{\text{po}}\rangle = \int_{V^2} dx\ dy\ \overline{\phi(x)}\psi(y)\cdot\delta(x-y) = \int_V \overline{\phi(x)}\psi(x)\ dx.$$

(b) We interpret a state ψ as two functions $\psi_{\text{po}}(q) = \langle q_{\text{po}}|\psi\rangle$ on V, and $\psi_{\text{mo}}(p) = \langle p_{\text{mo}}|\psi\rangle$ on V^*. They are related by

$$\psi_{\text{mo}}(p) = \langle p_{\text{mo}}|\psi\rangle = \langle p_{\text{mo}}| \int_V dx\ \psi_{\text{po}}(x)\cdot|x_{\text{po}}\rangle = \int_V dx\ \psi_{\text{po}}(x)\cdot\langle p_{\text{mo}}|x_{\text{po}}\rangle = \int_V dx\ \psi_{\text{po}}(x)\cdot e^{-\frac{i}{\hbar}p\cdot x}.$$

(c) We need A to make sense on the distributional states $\delta_x = |x_{\text{po}}\rangle$, then one should be able to pair $A\delta_x$ to δ_y, and finally, operator A should really be given by these infinitesimal contributions in the sense of

$$(A\psi)(y) \stackrel{\text{def}}{=} \langle y_{\text{po}}|A\psi\rangle = \ \langle\ y_{\text{po}}|A| \int dx\ \psi(x)\cdot|x_{\text{po}}\rangle\ \rangle = \int dx\ \psi(x)\cdot\langle y_{\text{po}}|A|x_{\text{po}}\rangle = \int dx\ \psi(x)\cdot A(x,y).$$

3.4.4. *Quantum view on Fourier analysis.* We have found a quantum view on $L^2(V)$ and on the Fourier transform. The Hilbert space $\mathcal{H}$ – the space of states – seems to be attached directly to the phase space T^*V.

However, for any Lagrangian vector subspace L such as V or V^*, $\mathcal{H}$ can be identified canonically with $L^2(L)$. Once we choose L, we get

- (i) a "distributional basis" $|a\rangle,\ a \in L$ of $\mathcal{H}$ (we also denote $|a\rangle$ by δ_a),
- (ii) an interpretation $\mathcal{H} \stackrel{\cong}{\to} L^2(L)$ by $\psi \mapsto \psi(a) \stackrel{\text{def}}{=} \langle \psi|\delta_a\rangle$, and
- (iii) the inner product on $\mathcal{H}$ given by the amplitudes can now be written in terms of this "distributional basis" as L^2-product written in the intriguing interaction form

$$\int_L \langle \phi|\delta_a\rangle\cdot\overline{\langle \delta_a|\phi\rangle}\ da.$$

3.4.5. *Hamiltonian operator $\widehat{H}$ and the Schroedinger's equation.* The time evolution $\psi(t)$ of a quantum state $\psi = \psi(0)$ is always given by the *abstract Schroedinger's equation*

$$\frac{d}{dt}\ \psi(t) = \ \frac{i}{\hbar}\widehat{H}\psi.$$

With the functional interpretation of states, a time dependent state $\psi(t)$ is a function of two variables $\psi(x,t)$, the Hamiltonian operator $\widehat{H}$ becomes an integral operator with the

kernel $H(x,y) \stackrel{\text{def}}{=} \langle \delta_x | \widehat{H} | \delta_y \rangle$, and the Schroedinger equation becomes an integral-differential equation

$$\frac{d}{dt}\ \psi(x,t)\ =\ \frac{i}{\hbar}\int\ \psi(x,t)H(x,y)\ dx.$$

For an electron on a line and in a potential $V(x)$, the Hamiltonian operator is

$$\widehat{H} = \ -\frac{\hbar^2}{2m}\ \frac{d^2}{dx^2} + V(x).$$

This is the quantization of the Hamiltonian function (the total energy)

$$H = \ \frac{p^2}{2m}\ + V(x).$$

Actually, this is the historical origin of the idea of a deformation quantization.

With this explicit Hamiltonian operator we get the original Schroedinger equation

$$i\hbar\frac{d}{dt}\ \psi(x,t)\ =\ -\frac{\hbar^2}{2m}\ \frac{d^2}{dx^2}\psi(x,t) + V(x)\psi(x).$$

By itself this equation describes (in principle!) all non-relativistic phenomena, such as energy levels in an atom and chemical bindings. The relativistic form was given by Dirac a year later (for instance it covers magnetism).

3.5. **Example: Harmonic oscillator.** Physicists think of the harmonic oscillator as a mathematical formalism which covers a number of physical situations. For instance a spring-mass-gravity system, a wave, or the above electron if a potential of the form $V(x) = \ c{\cdot}x^2$ with $c > 0$. Our goal will be to

understand the system in terms of the spectrum of the infinitesimal time generator $\widehat{H}$.

Our phase space will be the symplectic $\mathbb{R}^2$, i.e., $T^*\mathbb{R}$. However, we will find it convenient to think of it as a complex line $\mathbb{C}$ and use the holomorphic functions.

3.5.1. *The Hamiltonian function.* On the level of the classical equation of motion, harmonic oscillator is given by $mq'' = \ -kq$. In terms of the frequency parameter $\omega = \ \sqrt{k/m}$ this is

$$q'' + \omega^2 q = \ 0.$$

The Hamiltonian function is the total energy

$$H(p,q) = \ \frac{1}{2m}p^2\ +\ \frac{1}{2}kq^2.$$

If we choose the units so that we have the unit mass, this is $H(p,q) = \ \frac{1}{2}p^2\ +\ \frac{1}{2}\omega^2 q^2$.

3.5.2. *The quantization in standard coordinates.* We think of the phase space as the cotangent bundle $T^*\mathcal{R}$ to the q-line $\mathbb{R}_q$. Then the Hilbert space is naturally viewed as the space $\mathcal{H} = L^2(\mathbb{R}_q)$ of functions on the q-line. From this point of view the functions p, q on $T^*\mathbb{R}$ quantize to operators on $C^\infty(T^*\mathbb{R}) \subseteq \mathcal{H}$ that satisfy the unitarity prescription $[\widehat{p}, \widehat{q}] = \frac{\hbar}{i}\widehat{\{p, q\}}$,. So, $\{p, q\} = 1$ requires

$$[\widehat{p}, \widehat{q}] = \frac{\hbar}{i}.$$

This leads to the normalization

$$\widehat{p} \stackrel{\text{def}}{=} \frac{\hbar}{i}\frac{d}{dq}, \qquad \widehat{q} \stackrel{\text{def}}{=} q.$$

We will introduce the Hamiltonian operator $\widehat{H}$ as a naive quantization of the Hamiltonian function $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$, so

$$\widehat{H} = \frac{1}{2}\widehat{p}^2 + \frac{1}{2}\omega^2\widehat{q}^2.$$

Its eigenvectors in $L^2(\mathbb{R}_q)$ are the Hermite polynomials.

3.5.3. *The quantization in holomorphic coordinates.* We will pass to holomorphic coordinates on $T^*\mathbb{R} = \mathbb{R}^2$ given by

$$\begin{array}{rcl} a &=& \frac{1}{\sqrt{2\omega}}(\omega q + ip), \\ a^* &=& \frac{1}{\sqrt{2\omega}}(\omega q - ip), \end{array} \quad \text{and} \quad \begin{array}{rcl} q &=& \frac{1}{\sqrt{2\omega}}(a + a^*), \\ p &=& \frac{\sqrt{\omega/2}}{i}(a - a^*). \end{array}$$

Here, a^* is the complex conjugate of the function a. We will think of a^* as the basic holomorphic coordinate z.

3.5.4. *Lemma.* (a) The Poisson structure $\{p, q\} = 1$, is in new coordinates

$$\{a, a^*\} = i.$$

(b) The quantizations of the generators are operators satisfying the commutation relation from $D_{\mathbb{A}^1}$

$$[\widehat{a}, \widehat{a}^*] = \hbar.$$

(c) At $\hbar = 1$ one can realize the algebra generated by $\widehat{a}, \widehat{a}^*$ and the above relation, as the algebra of polynomial differential operators on holomorphic functions in z, by

$$a^* = z \quad \text{and} \quad a = \hbar\partial_z = \partial.$$

(d) The Hilbert space $\mathcal{H}$ is realized as the Hilbert space of entire functions in $z \stackrel{\text{def}}{=} a^*$. It is a completion of the space of polynomials $\mathbb{C}[z] = \mathbb{C}[a^*]$ with respect to the inner product

$$(f, g) \stackrel{\text{def}}{=} \int_{\mathbb{C}} \overline{f(z)}\, g(z)\, e^{-z\bar{z}}\, \frac{dzd\bar{z}}{2\pi i} = \int_{\mathbb{C}} \overline{f(a^*)}\, g(a^*)\, e^{-a^*a}\, \frac{da^*da}{2\pi i}.$$

Proof. (a) The Poisson structure $\{p,q\} = 1$ is now

$$\{a,a^*\} = \{\frac{\omega q+ip}{\sqrt{2\omega}},\frac{\omega q-ip}{\sqrt{2\omega}}\} = \frac{1}{2\omega}\{\omega q+ip,\omega q-ip\} = \frac{1}{2\omega}2\omega i\{p,q\} = i.$$

(b) The quantizations of the generators are operators satisfying the commutation relation

$$[\widehat{a},\widehat{a}^*] = \frac{i}{\hbar}\{a,a^*\} = \frac{i}{\hbar}\cdot\hbar.$$

(c) For the above operators

$$[\widehat{a},\widehat{a}^*] = [\frac{1}{\sqrt{2\omega}}(\omega\widehat{q}+i\widehat{p}),\frac{1}{\sqrt{2\omega}}(\omega\widehat{q}-i\widehat{p})] = \frac{1}{2\omega}2\omega i[\widehat{p},\widehat{q}] = i\frac{\hbar}{i} = \hbar.$$

3.5.5. *Corollary.* (a) The Hamiltonian operator is

$$\widehat{H} = \omega\,\widehat{a}^*\cdot\widehat{a} = \hbar\omega\; z\partial.$$

(b) The eigenvectors of $\widehat{H}$ are the powers of $z = a^*$ [16]. We normalize them to unit vectors $\psi_n \overset{\text{def}}{=} \frac{(a^*)^n}{\sqrt{n!}} = \frac{z^n}{\sqrt{n!}}$. Then

$$\widehat{H}\psi_n = \hbar\omega(n+\frac{1}{2})\cdot\psi_n.$$

The physical meaning is that the energy of the state ψ_n is $\hbar\omega(n+\frac{1}{2})$.

Proof. (a) We just recalculate the Hamiltonian from standard coordinates

$$\widehat{H} = \frac{1}{2}\widehat{p}^2 + \frac{1}{2}\omega^2\widehat{q}^2 = \frac{1}{2}[\frac{\sqrt{\omega/2}}{i}(\widehat{a}-\widehat{a}^*)]^2 + \frac{1}{2}\omega^2[\frac{1}{\sqrt{2\omega}}(\widehat{a}+\widehat{a}^*)]^2$$

$$= \frac{-\omega}{4}[\widehat{a}^2+(\widehat{a}^*)^2-\widehat{a}\widehat{a}^*-\widehat{a}^*\widehat{a}] + \frac{\omega}{4}[\widehat{a}^2+(\widehat{a}^*)^2+\widehat{a}\widehat{a}^*+\widehat{a}^*\widehat{a}] = \frac{\omega}{2}(\widehat{a}\widehat{a}^* + \widehat{a}^*\widehat{a})$$

$$= \omega(\widehat{a}^*\widehat{a}+\frac{\hbar}{2}) = \omega(\widehat{a}\widehat{a}^*-\frac{\hbar}{2}) = \hbar\omega(z\partial+\frac{1}{2}).$$

(b) Now,

$$\widehat{H}z^n = \hbar\omega(z\partial+\frac{1}{2})z^n = \hbar\omega(n+\frac{1}{2})\cdot z^n.$$

The change of variables $z = x+iy = re^{i\theta}$, gives $dzd\bar{z} = (dx+idy)(dx-idy) = 2i\;dy\;dx = 2ir\;dr\;d\theta$, hence

$$= \langle\psi_n,\psi_m\rangle = \int_{\mathbb{C}}\frac{\overline{z^n}}{\sqrt{n!}}\,\frac{z^m}{\sqrt{m!}}\cdot e^{-z\bar{z}}\,\frac{dzd\bar{z}}{2\pi i} = \int_{r=0}^{+\infty}\int_{\theta=0}^{2\pi}\frac{r^{n+m}}{\sqrt{n!}\sqrt{m!}}e^{i\theta(m-n)}\cdot e^{-r^2}\,\frac{2ir\;dr\;d\theta}{2\pi i}$$

$$= \delta_{nm}\cdot\int_{r=0}^{+\infty}\frac{r^{2n}}{n!}\cdot e^{-r^2}\;2rdr = \delta_{nm}\cdot\int_{u=0}^{+\infty}\frac{u^n}{n!}\cdot e^{-u}\;du$$

[16] better then the special functions!

3.5.6. *The energy spectrum of a harmonic oscillator.* The *energy spectrum* of the theory is the set of eigenvalues of the Hamiltonian operator. We found that the energy spectrum of a harmonic oscillator with frequency $\omega = \sqrt{k/m}$ is[17]

$$\hbar\omega(n+\frac{1}{2}), \quad n=0,1,... \qquad i.e., \quad \frac{\omega\hbar}{2}, \frac{3\omega\hbar}{2}, \frac{5\omega\hbar}{2}, ...$$

3.5.7. *The representation theoretic organization of the Hilbert space* (*language of creation and annihilation*). Instead of considering a particular realization of the Hilbert space of states $\mathcal{H}$, one often considers the way it is organized by the action of the quantized algebra of observables, i.e., an algebraic representation of the algebra of observables on a dense vector subspace V of $\mathcal{H}$.

The action of the algebra $D_{\mathbb{A}^1}(\hbar) = \oplus\, \mathbb{C}\widehat{p}^i\widehat{q}^j = \oplus\, \mathbb{C}\widehat{a}^i\widehat{a^*}^j$ on $\mathcal{H}$ is given by the action of two operators a and a^*. The space is graded by the action of the Hamiltonian operator $\widehat{H}$ which corresponds to energy. We have observed that

Lemma. (a) Eigenvalues of $\widehat{H}$ are bounded from bellow and there is a unique vector (line) on which it has the smallest eigenvalue. We call it the vacum vector $|_{\rm vac}\rangle$.

(b) The vacum is killed by a and applying a^* to the vacum creates a bases $(a^*)^n|_{\rm vac}\rangle,\ n \geq 0$ of $\mathcal{H}$.

So we call a^* a creation operator, and a the annihilation operator, since a kills the vacum and each vector is killed by some power of a. From this algebraic point of view, the interesting space is the subspace $V = \oplus_{n\geq 0}\cdot(a^*)^n|_{\rm vac}\rangle \subseteq \mathcal{H}$.

3.5.8. *Planck constant.* To simplify formulas one often uses a choice of units for which $\hbar = 1$. However, one will sometimes think of $\hbar$ as a variable. *Physically* it corresponds to the idea that when we *change the scale* that we are interested in, our view of the size of $\hbar$ changes. *Formally*, it is a useful trick to have an *additional variable.*

3.5.9. *Harmonic oscillator in a metric vector space.* We consider a real vector space M with a metric G which we diagonalize into

$$G = \sum (x^\mu)^2 - \sum (y^\nu)^2.$$

[17]We have calculated with the unit mass, but this did not affect the spectrum.

4. **Path integrals: the Lagrangian approach**

Feynman's idea of a path integral formula for the amplitude of an event is that one integrates over all possible evolutions (*histories*) of the system that result in this event. This is said to be the most physically fundamental and useful point of view. However, we will follow the historical development and deduce the path integral formula as a consequence of the operator approach.

4.0.10. *Theorem. One can write this amplitude as a Feynman integral over paths in the tangent bundle*[18]

$$\langle x_f, t_f | x_i, t_i \rangle = \int_{(x,y):\ [t_i,t_f]\to T\mathcal{C},\ x(t_i)=x_i,\ x(t_f)=x_f} d(x,y) \quad e^{\frac{i}{\hbar}\mathcal{S}[x,y]} \ .$$

Here, one uses the action on such paths

$$\mathcal{S}[x,y] \stackrel{def}{=} \int_{t_i}^{t_f} dt\ L(x(t),\dot{x}(t),y(t)) = \int_{t_i}^{t_f} dt\ \frac{1}{2}\dot{x}\cdot p - \ V(x),$$

which for velocity curves $(x,\dot{x})$ reduces to the standard action $S[x]$.

In the remainder we check this claim. What we get is rather a formula involving a limit over some N-step piecewise curves

$$\lim_{N\to\infty} \int_{\text{“}N\text{-curves”}\ (x,p)_{(t_*,x_*,p_*)}\ \text{in}\ T^*\mathcal{C}} d(x,p)_{(t_*,x_*,p_*)} \quad e^{\frac{i}{\hbar}\mathcal{S}[(x,p)_{(t_*,x_*,p_*)}]} \ ,$$

The path integral formula in the theorem is an interpretation (or shorthand) for this.

4.0.11. *The setting for the formula.* We consider a quantum mechanical system in a Hamiltonian formalism, and we calculate the amplitude that the system evolves from the initial pure position state x_i at the initial time t_i, to the final pure position state x_f at the final time t_f. This is a matrix coefficient of the evolution operator $U(\Delta t) = \ e^{\frac{i}{\hbar}\cdot\Delta t\cdot\widehat{H}}$,

$$\langle x_f, t_f | x_i, t_i \rangle = \ \langle x_f | U(t_f - t_i) | x_i \rangle.$$

We will write it in the form of a *Feynman integral* over all paths into $T\mathcal{C}$ (or $T^*\mathcal{C}$).

4.0.12. *The appearance of the Feynman integral.* It is obtained by trying to account for the amplitude that x_i at time t_i will evolve to x_f at time t_f, as an integral over the time interval $[t_i, t_f]$, i.e., as an accumulation of contributions over small time subintervals $[t_{k-1}, t_k]$.

(1) Inserting points $t_1, ..., t_{N-1}$ between t_i and t_f leads to an integral $\int_{\mathbb{R}^{N-1}} dx_1 \cdots dx_{N-1}$ over all possible positions x_k at moments t_k.

[18] Hopefully (?) this is an example of the philosophy of adding an auxiliary field to the action from I.3.4.1. Here, one is adding y to x.

(2) For short time intervals $[t_{k-1}, t_k]$, the linear approximation (in $t_k - t_{k-1}$), of the amplitude is given by an integral $\int_{\mathbb{R}^N} dp_1 \cdot \cdots dp_N$ of contributions from all possible momenta p_k.
(3) So, an approximation of the amplitude based on the partition $t_i = t_0, ..., t_N = t_f$, ends up being an integral over N possible positions (x_k, p_k) in $T^*\mathcal{C}$.
(4) Interestingly, the integrand is exactly $e^{\frac{i}{\hbar}\mathcal{S}}$ for an obvious action $\mathcal{S}$ of the piecewise liner path in $T^*\mathcal{C}$ given by these points. So we get the Feynman integral over all "N-step curves" in $T^*\mathcal{C}$.
(5) Finally, as N gets large we get the precise formula for the amplitude – and we interpret the limit of N-step curves to be the moduli of all curves in $T^*\mathcal{C}$.

4.1. **Matrix coefficients of $\widehat{f}$ for the normal ordering quantization.** Recall that for a state ψ we denote by

$$|\psi, t\rangle = U(t^{-1})|\psi\rangle,$$

the "state $\psi\rangle$ at time t", i.e., the state which in the period from time 0 to time t evolves to $\psi\rangle$.

4.1.1. *Problem.* Find the amplitude that the system evolves from the initial pure position state x_i at the initial time t_i, to the final pure position state x_f at the final time t_f. This is the matrix coefficient

$$\langle x_f, t_f | x_i, t_i\rangle = \langle U(-t_f)x_f | U(-t_i)x_i\rangle = \langle x_f | U(t_f - t_i)|x_i\rangle.$$

4.1.2. *Assumptions on the physical system.* We will do all calculations for a system with the Hamiltonian function and the Hamiltonian operator

$$H = \frac{p^2}{2m} + V(x) \quad \text{and} \quad \widehat{H} = \frac{\widehat{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\partial_x^2 + V(x).$$

Say, for an electron on a line and in a potential V.

The Hilbert space $\mathcal{H}$ has distributional basis given by pure position states $|a_{\text{po}}\rangle$, $a \in \mathbb{R}$, and the pure momentum states $|b_{\text{mo}}\rangle$, $b \in \mathbb{R}$. The algebra of operators on $\mathcal{H}$ is generated by the position and momentum operators $\widehat{x} = x$ and $\widehat{p} = \frac{i}{\hbar}\partial_x$. They act by

$$\widehat{x}|a_{\text{po}}\rangle = a\cdot|a_{\text{po}}\rangle, \qquad \widehat{p}|b_{\text{mo}}\rangle = b\cdot|b_{\text{mo}}\rangle.$$

i.e., in terms of the functional interpretation:

$$x\cdot\delta_a = a\cdot\delta_a \quad \text{and} \quad \frac{\hbar}{i}\partial_x\, e^{\frac{i}{\hbar}xb} = b\cdot e^{\frac{i}{\hbar}xb}.$$

4.1.3. *Normal ordering and Weyl ordering.* In general the Hamiltonian operator is obtained from the Hamiltonian function by a (purely defined) quantization procedure which requires some choices. Here we notice that the above Hamiltonian function is simple enough, so that its quantization is independent of the choices.

Recall that the identification of functions on $T^*\mathcal{C}$ and $D_{\mathcal{C}}$ is not canonical - it is only defined canonically on the generators x and p. The way one assigns operators to functions, $f \mapsto \widehat{f}$ is therefore not canonical, so the choice of $\widehat{H}$ is not canonical in general.

The simplest identification is obtained by $p^i q^j \leftrightarrow \widehat{p}^i \widehat{q}^j$, hence $\widehat{(p^i q^j)} = \widehat{p}^i \widehat{q}^j$. This prescription can be described by "p before q", or "annihilation before creation". It is called *normal ordering.*

Another useful choice is the *Weyl ordering* which chooses symmetric expressions, say,

$$\widehat{pq} = \frac{1}{2}(\widehat{p}\widehat{q} + \widehat{q}\widehat{p}).$$

All such prescriptions work the same on the Hamiltonian function if does not mix the variables, i.e $H = H_{kinetic}(p) + H_{potential}(x)$, as we have assumed above.

4.1.4. *Lemma.* For $f \in \mathbb{C}[x,p] = \mathbb{C}[q,p]$

$$(a) \quad \langle p_{\mathrm{mo}} | \widehat{f} | x_{\mathrm{po}} \rangle = e^{-\frac{i}{\hbar} px} \cdot f(x,p),$$

$$(b) \quad \langle y_{\mathrm{po}} | \widehat{f} | x_{\mathrm{po}} \rangle = \int_{\mathbb{R}} dp \; e^{\frac{i}{\hbar} p(y-x)} \cdot f(x,p).$$

$$(c) \quad \langle y_{\mathrm{po}} | e^{s\widehat{f}} | x_{\mathrm{po}} \rangle = \int_{\mathbb{R}} dp \; e^{\frac{i}{\hbar} p(y-x) + sf(x,p)} \quad (mod\ s^2).$$

Proof. (a) It suffices to check (a) for $f = p^i x^j$. For the normal ordering quantization the LHS is $\langle p_{\mathrm{mo}} | \widehat{p}^i \widehat{x}^j | x_{\mathrm{po}} \rangle = \langle p_{\mathrm{mo}} | x_{\mathrm{po}} \rangle \cdot p^i x^j = e^{-\frac{i}{\hbar} px} \cdot f(x,p)$. Now (b) follows

$$\langle y_{\mathrm{po}} | \widehat{f} | x_{\mathrm{po}} \rangle = \int_{\mathbb{R}} dp \; \langle y_{\mathrm{po}} | p_{\mathrm{mo}} \rangle \cdot \langle p_{\mathrm{mo}} | \widehat{f} | x_{\mathrm{po}} \rangle = \int_{\mathbb{R}} dp \; \overline{e^{-\frac{i}{\hbar} py}} \cdot e^{-\frac{i}{\hbar} px} \cdot f(x,p).$$

In (c) the claim is that the RHS is the linearization of the LHS. At $s = 0$ both sides equal $\delta(x-y)$, applying $\frac{d}{ds}|_{s=0}$ to (c) yields known claim (b).

4.1.5. *Remark.* (1) Our formal calculations are meaningful in the appropriate spaces of distributions. For instance, the integral in (b) does not converge absolutely. The only meaning of this integral is that it is a distribution on the xy-plane, hence absolute convergence holds only for integrals

$$\int_{\mathbb{R}^2} dx \; dy \; \phi(x,y) \cdot \langle y_{\mathrm{po}} | \widehat{H} | x_{\mathrm{po}} \rangle = \int_{\mathbb{R}^3} dx \; dy \; dp \; \phi(x,y) \cdot e^{\frac{i}{\hbar} p(y-x)} \cdot H(x,p)$$

where function ϕ is in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$.

(2) Claim (c) says that the the short time evolution of pure position states equals, modulo $(t''-t')^2$,

$$\langle x''_{\rm po}, t''|x'_{\rm po}, t'\rangle = \langle x''_{\rm po}|U(t''-t')|x'_{\rm po}\rangle = \langle x''_{\rm po}|e^{-\frac{i}{\hbar}(t''-t')\widehat{H}}|x'_{\rm po}\rangle$$
$$= \int_{\mathbb{R}} dp \;\; e^{\frac{i}{\hbar}[p(x''-x')-(t''-t')H(x,p)]}.$$

4.2. **Localization of the amplitude on the time interval.**

4.2.1. *Insertion of a point in the time interval.* For times $a<b<c$ and positions x, z, the formula

$$\langle z, c|x, a\rangle = \int_{\mathbb{R}} dy \; \langle z, c|y, b\rangle\cdot\langle y, b|x, a\rangle,$$

can be justified on the level of physics (any evolution from x at time a to z at time c has to go through some position y at time b), or mathematics (the states $|y, b\rangle$, $y\in\mathbb{R}$, form a distributional basis of $\mathcal{H}$).

4.2.2. *Amplitude as an integral over possible intermediate positions.* Now we insert $N-1$ points into the time interval: $t_i = t_0 < t_1 < \cdots < t_N = t_f$ and we denote $x_0 = x_i$, $x_N = x_f$. Then

$$\langle x_f, t_f|x_i, t_i\rangle = \int_{\mathbb{R}^{N-1}} dx_1\cdots dx_{N-1} \prod_{1\le k\le N}\langle x_k, t_k|x_{k-1}t_{k-1}\rangle.$$

(There are more factors in the product then variables of integration.)

4.2.3. *Amplitude as an integral over possible intermediate positions and momenta.* Suppose that the intervals are of the same length: $t_k - t_{k-1} = \varepsilon \overset{\rm def}{=} \frac{t_f-t_i}{N}$. According to the linearization formula of the time evolution (4.1.5.2), *modulo* ε^2 we have

$$\langle x_k, t_k|x_{k-1}t_{k-1}\rangle = \int_{\mathbb{R}} dp_k \; e^{\frac{i}{\hbar}[p_k(x_k-x_{k-1})-\varepsilon\cdot H(x_{k-1},p_k)]}.$$

Therefore, *modulo* ε

$$\langle x_f, t_f|x_i, t_i\rangle = \int_{\mathbb{R}} dx_1\cdots dx_{N-1} \int_{\mathbb{R}^N} dp_1\cdots dp_N \; e^{\frac{i}{\hbar}[\sum_1^N \; p_k(x_k-x_{k-1})-\varepsilon\cdot H(x_{k-1},p_k)]}$$

(since $\prod_1^N \; (c_i+\varepsilon^2) = \prod_1^N \; c_i + \varepsilon^2\sum_1^N \; \prod_{j\neq i} c_j + \cdots$, and $\varepsilon^2\sum_1^N$ behaves like 1).

4.2.4. *Interpretations.* The integral over all choices of k positions and k momenta will be interpreted as an integral over all curves (x, p) in $T^*\mathcal{C}$. Then the phase of the integrand will be viewed as an action for curves in $T^*\mathcal{C}$.

4.2.5. *An action for paths in the (co)tangent bundle.* In a traditional manner we start with paths $(x,y):\ [t_i,t_f]\to T\mathcal{C}$ into the tangent bundle. For these we choose a version of the usual Lagrangian function on $T\mathcal{C}$ as

$$\mathcal{L}(x,\dot{x},y)=\ T(\dot{x},y)-V(x)=\ \frac{m}{2}\dot{x}\cdot y-\ V(x).$$

When we will translate this into a path (x,p) in the phase space $T^*\mathcal{C}$, (using $p=ym$), we get a version of the usual Hamiltonian function

$$\mathcal{H}(x,\dot{x},p)=\ \dot{x}\cdot p-L(x,\dot{x},y)=\ \frac{1}{2}\dot{x}\cdot p+\ V(x).$$

Observe that for velocity and momentum curves (i.e., $y=\dot{x}$), these are the usual Lagrangian and Hamiltonian of a path x in $\mathcal{C}$.[19]

The corresponding action for paths in $T\mathcal{C}$ (or $T^*\mathcal{C}$) is

$$\mathcal{S}(x,y)\stackrel{\text{def}}{=}\int_{t_i}^{t_f}dt\ L(x(t),\dot{x}(t),y(t))=\int_{t_i}^{t_f}dt\ \frac{1}{2}\dot{x}\cdot p-V(x)=\int_{t_i}^{t_f}dt\ \dot{x}\cdot p-H(x,\dot{x},p)=\ \mathcal{S}(x,p)$$

4.2.6. *The integrand and the piecewise linear curves in $T^*\mathcal{C}$.* We can use the data $(t_*,x_*,p_*)=\ (t_0,...,t_N,x_0,...,x_N,p_1,...,p_N)$, ti cook up a piecewise linear (discontinuous) curve $(x,p)_{(t_*,x_*,p_*)}:\ [t_i,t_f]\to T^*\mathcal{C}$, such that for $t_{k-1}\le t<t_k$,

$$\begin{array}{c}x(t)\text{ is a line from }x_{k-1}\text{ to }x_k,\text{ and}\\ \text{and }p(t)\text{ is a constant }p_k.\end{array}$$

We will call these curves the N-curves, and think of them as the level N approximation of the moduli of curves in $T^*\mathcal{C}$. Then the phase of the integrand is the action of this N-curve

$$phase=\ \sum_1^N\ p_k(x_k-x_{k-1})-\varepsilon\cdot H(x_{k-1},p_k)]=\ \sum_1^N\ (t_k-t_{k-1})\cdot[p_k\frac{x_k-x_{k-1}}{t_k-t_{k-1}}-H(x_{k-1},p_k)]$$

$$=\ \mathcal{S}[(x,p)_{(t_*,x_*,p_*)}]$$

If p would coincide with $m\cdot\dot{x}$ this would be the formula for the action, en

4.2.7. *The limit of the integrand.* For a curve (x,p) in $T^*\mathcal{C}$, we define x_k,p_k as values at t_k, and we get a piecewise linear discontinuous approximation $(x,p)_{(t_*,x_*,p_*)}$ of the curve (x,p). We notice that the action is continuous in these approximations, i.e.,

$$\lim_{N\to\infty}\ \mathcal{S}[(x,p)_{(t_*,x_*,p_*)}]=\ \lim_{N\to\infty}\ \sum_1^N\ (t_k-t_{k-1})\cdot[p_k\frac{x_k-x_{k-1}}{t_k-t_{k-1}}-H(x_{k-1},p_k)]$$

$$=\ \int_{t_i}^{t_f}\ dt\ p(t)\cdot\dot{x}(t)-H(x(t),p(t))=\ \mathcal{S}[x,p].$$

[19]However, our $\mathcal{L}$ and $\mathcal{H}$ are functions on $T\mathcal{C}\times_{\mathcal{C}}T\mathcal{C}$ and $T^*\mathcal{C}\times_{\mathcal{C}}T^*\mathcal{C}$, rather then $T\mathcal{C}$ and $T^*\mathcal{C}$.

4.2.8. Now we have interpreted the approximate (modulo $\varepsilon = (t_f - t_i)/N$), calculation of the amplitude $\langle x_f, t_f | x_i, t_i \rangle$ as the integral over all "N-curves" in $T^*\mathcal{C}$

$$\langle x_f, t_f | x_i, t_i \rangle \overset{\sim}{=} \lim_{N\to\infty} \int_{\mathbb{R}^{N-1}} dx_1 \cdots dx_{N-1} \int_{\mathbb{R}^N} dp_1 \cdots dp_N \, e^{\frac{i}{\hbar}[\sum_1^N p_k(x_k - x_{k-1}) - \varepsilon \cdot H(x_{k-1}, p_k)]}$$

$$= \lim_{N\to\infty} \int_{\text{"}N\text{-curves" } (x,p)_{(t_*, x_*, p_*)} \text{ in } T^*\mathcal{C}} d(x,p)_{(t_*,x_*,p_*)} \quad e^{\frac{i}{\hbar}\mathcal{S}[(x,p)_{(t_*,x_*,p_*)}]} \, .$$

Finally, when we pass to the limit $N \to \infty$, we get an exact formula for the amplitude

$$\langle x_f, t_f | x_i, t_i \rangle = \lim_{N\to\infty} \int_{\text{"}N\text{-curves" } (x,p)_{(t_*, x_*, p_*)} \text{ in } T^*\mathcal{C}} d_{(t_*,x_*,p_*)} \quad e^{\frac{i}{\hbar}\mathcal{S}[(x,p)_{(t_*,x_*,p_*)}]} \, .$$

To interpret it we imagine that the moduli of N-curves should converge to the moduli of all curves in $T^*\mathcal{C}$. This gives

$$\langle x_f, t_f | x_i, t_i \rangle = \int_{(x,p):\ [t_i,t_f]\to T^*\mathcal{C},\ x(t_i)=x_i,\ x(t_f)=x_f} d(x,p) \quad e^{\frac{i}{\hbar}\mathcal{S}[x,p]}$$

$$= \int_{(x,y):\ [t_i,t_f]\to T\mathcal{C},\ x(t_i)=x_i,\ x(t_f)=x_f} d(x,y) \quad e^{\frac{i}{\hbar}\mathcal{S}[x,y]} \, .$$

4.2.9. *Remark.* The action here involves curves (x, y) into $T\mathcal{C}$ (the same as (x, p) into $T^*\mathcal{C}$), but this is different then just the velocity curves of curves into $\mathcal{C}$.

5. **Example:** Relativistic free particle

Here we repeat the calculation of amplitude for a free particle, except that in the relativistic setting we work in a larger space with auxiliary fields: an arbitrary parameterization x (rather then the parameterization by time), and a worldsheet metric g. In terms of the analogy with strings we are now working with the (relativistic!) Polyakov-type action for particles.

[20]

5.1. **Amplitude calculation.** For two events[21] x_i, x_f in the *spacetime*, the amplitude of x_i evolving to x_f is

$$\langle x_f | x_i \rangle = N \cdot \int_{x_i}^{x_f} Dx \int_{metrics\ g} Dg \quad e^{\frac{i}{\hbar} S[x,g]}.$$

Here N is an unknown constant. The notation in the first integral means that we fix an interval $I = [\tau_i, \tau_f]$ and we integrate over all paths $x : I \to M$ with the boundary conditions $x(\tau_i) = x_i$ and $x(\tau_f) = x_f$.

5.2. **Gauge fixing (= Equivariant integration).** Consider an integral $\int_X f$ with a symmetry group Γ, i.e., Γ acts on X and preserves the function f and the measure on X. Then one can try to define the equivariant integral as

$$\int_X^\Gamma f \stackrel{\text{def}}{=} \int_{\Gamma\backslash X} \overline{f},$$

where $\overline{f}$ is the descent of f to the quotient X/Γ (and one uses the quotient measure on X/Γ). Actually, if the symmetry is infinite, this is the only meaning one can attach to the original integral $\int_X f$.

Next, if one knows a section $s : \Gamma\backslash X \to X$ of the quotient map (in physics *section* is called *gauge*), then one can rewrite the integral on the quotient as an integral $\int_{s(\Gamma\backslash X)} f$ on the section $s(\Gamma\backslash X)$ which is a subspace of X, hence less abstract.

5.2.1. *Problems.* One needs

(1) a "good" quotient $\Gamma\backslash X$,
(2) the correct measure (the *Faddeev-Popov measure*), on the section $s(\Gamma\backslash X)$.

[20] FIND SOURCE OR SIMPLIFY THE STRING VERSION

[21] In spacetime we will use terminology *event* rather then *position*, to emphasize that the difference from the space position.

5.2.2. *The standard example.* Let $\Gamma = \mathbb{Z}$ act on $X = \mathbb{R}$ by translations and consider an invariant (i.e., periodic) function $f(x+n) = f(x),\ x \in \mathbb{R}, n \in \mathbb{Z}$. Now the $\mathbb{Z}$-equivariant integration of a periodic function on $\mathcal{R}$ is just the integration on the circle $\Gamma\backslash\mathbb{R} = S^1$

$$\int_{\mathbb{R}}^{\Gamma} dx\ f(x) \stackrel{\text{def}}{=} \int_{S^1} dx\ \overline{f}(e^{2\pi i x}) = \int_0^1 dx\ f(x),$$

and in the last equality we use the standard (discontinuous) section $S^1 \to [0,1) \subseteq \mathbb{R}$.

5.3. **Gauge fixing of the metric g and length as a Teichmüller parameter (left-over parameter).**

5.4. **Amplitudes.**

Part III. (Quantum) Field Theory

0. Intro

Text. Borcherds, *Quantum Field Theory.*

Quantum Field Theory is an extension of the formalism of Quantum Mechanics from particles to fields.

0.0.1. *Field theories.* Mechanics deals with a particle like objects that are supported at a point of the spacetime. At this point a particle information (mass, velocity, charge,...) is a a vector.

Field theories deal with objects ("fields") that are distributed over some space. So, a field is something like a function on a manifold. More generally fields are distributions with values in a vector bundle (or something like that).

The most obvious are the fields on the spacetime, say, the electric fields, etc. However, there are also fields on the worldsheet, say, a metric on the worldsheet.

0.0.2. *σ-models.* These are the worldsheet field theories where the main field is the evolution map x (or its component functions), viewed as a function on the worldsheet Σ with values in the spacetime (M, G), i.e., a manifold M with a metric G (which need not be positive definite).

So mathematically this is the science of maps between manifolds. The ones of physical interest are the ones that minimize certain action such as (i) volume or (ii) energy. In mathematics such maps are called (i) minimal submanifolds, and (ii) harmonic maps.

The terminology *σ-model* comes simply from the habit of denoting the source of the map (the worldsheet) by Σ.

So, classical mechanics can be viewed as a σ-model worldline field theory

0.0.3. *Lagrangian view: Feynman's path integrals.* The key notion is the action $S[x]$. A classical theory formulates physical laws by:

$$\text{evolution } x \text{ is a solution of the criticality equation, i.e., } d_x S = 0.$$

A quantum theory in ...

0.0.4. *Calculus of variations.* This is the mathematical framework for the classical Lagrangian approach, i.e., in classical physics the evolution is a solution of an EL-equation, we say that it is a *classical solution.*

The *classical worldsheet field theories* (i.e., *classical σ-models*), for strings and branes will require calculus of variations for any worldsheet manifold Σ. (See I.1.3 for the case $\dim(\Sigma) = 1$.)

1. Classical Field Theory: Calculus of Variations

1.0.5. *Goals.* Classically, physical laws are expressed in the Lagrangian approach by requiring that the certain quantity $S[x]$ attached to a possible evolution x of the system is minimal. One calls $S[x]$ the action of x, the criticality equation $d_x S = 0$ is called the Euler-Lagrangian equation, and from the quantum point of view the solutions of the EL-equation are called the *classical solutions.*

Bellow we calculate the EL-equation in a standard special setting and in local coordinates.

1.0.6. *The setting.* Here, the field x is a map $x : \mathbf{\Sigma} \to M$ from a manifold (with boundary) $\mathbf{\Sigma}$ to a manifold M. The action $S[x]$ is an integral

$$S[x] \stackrel{\text{def}}{=} \int_{\mathbf{\Sigma}} d\tau \ \boldsymbol{L}(x(\tau), \dot{x}(\tau), \tau) \ ,$$

of a function $\boldsymbol{L}(x, v, \tau)$ which is a (possibly $\mathbf{\Sigma}$-dependent) function of the position x in M, and velocity $v \in T_x M$. So the *Lagrangian* $\boldsymbol{L}(x, v, \tau)$ is a function on $TM \times \mathbf{\Sigma}$.

1.0.7. *Reduction to the standard* **local** *setting.* The criticality equation will be local in $\mathbf{\Sigma}$, and we calculate it in local coordinates τ^i on an open piece $\Sigma \subseteq \mathbf{\Sigma}$, which we identify via the coordinates τ^i with a coordinate box $\Sigma \subseteq \mathbb{R}^p$. In the process, the measure $d\tau$ on $\mathbf{\Sigma}$ becomes a multiple $\mu(\tau) \cdot d\tau^0 \cdot \cdots d\tau^p$ of the standard measure on $\Sigma \subseteq \mathbb{R}^p$, and we incorporate it into a new Lagrangian $\boldsymbol{L}(x, v, \tau) = \ \mu(\tau) \cdot L(x, v, \tau)$. We do all our calculations in the setting of a box Σ and the corresponding Lagrangian $L = \mu \cdot \boldsymbol{L}$.

1.0.8. *EL-equations.* In the above local coordinates the criticality equation becomes

(1) on Σ, there are ingredients L_x and $L_{\frac{dx}{d\tau}}$ that come from the non-dynamical dependence on the field x (dependence of L on x, no Σ-derivatives), and the dynamical dependence on x (dependence of L on the Σ-derivatives of x):

$$L_x \ = \ \frac{d}{d\tau}(L_{\frac{dx}{d\tau}}),$$

(2) on $\partial\Sigma$ we have the Von Neumann Boundary condition

$$L_{\frac{\partial x}{\partial n}} = 0.$$

1.0.9. *Mathematical analysis.* Here we really do calculus in topological vector spaces, but I will omit the relevant technology.

1.1. Variation $\frac{\delta S}{\delta x}$ and pointwise variation $\frac{\delta S}{\delta x(t)}$ of the action.

1.1.1. *Action.* We are interested in the critical points of a functional $S[x]$ on the space of all paths $x : \Sigma \to M$ in a manifold M. We will call $S[x]$ the *action* of the path x, it will be an integral

$$S[x] \stackrel{\text{def}}{=} \int_\Sigma d\tau\ L(x(\tau), \dot{x}(\tau), \tau)\ ,$$

of a function $L(x, v, \tau)$ which is a (possibly time dependent) function of the position x in M, and velocity $v \in T_xM$. So the *Lagrangian* $L(x, v, \tau)$ is a function on $TM \times \Sigma$.

1.1.2. *Variation* $\frac{\delta S}{\delta x}$ *and pointwise variation* $\frac{\delta S}{\delta x(t)}$. The variation means just the differential $\frac{\delta S}{\delta x} \stackrel{\text{def}}{=} d_x S \ \in\ T^*_x[C^\infty(\Sigma, M)$.

Its distributional version is the variation with respect to the value of x at a fixed point $t \in \Sigma$. For a tangent vector $v \in\ T_{x(t)}M$ we consider the variation of $S[x]$ when one moves x by a distributional path $u = \delta_t(\tau)\cdot v$ concentrated at the point $t \in \Sigma$, so

$$\frac{\delta S}{\delta x(t)} v \ \stackrel{\text{def}}{=}\ \frac{\delta S}{\delta x}\ \delta_t(\tau)\cdot v \ =\ (d_x S)\ \delta_t(\tau)\cdot v.$$

1.1.3. *Local situation.* For simplicity we will calculate in the local situation, so Σ will be a rectangle $\prod_1^p\ I_k \subseteq \mathbb{R}^p$ for some intervals $I_k = \ \{a_i \le \tau^i \le b_i\}$, and the target M will be a vector space. Now

$$S[x] = \int_\Sigma\ dvol(\tau)\ L(x, \dot{x}, \tau) = \int_{I_1} \cdots \int_{I_p} d\tau^1 \cdots d\tau^p\ L(x(\tau), \dot{x}(\tau), \tau).$$

The dot denotes the derivative with respect to τ. So, $\dot{x}(\tau) : T_\tau\Sigma \to T_{x(\tau)}M$, i.e., $\dot{x}$ is a section of x^*TM and in local coordinates it is a matrix $\frac{\delta x^\mu}{\delta \tau_k}$.

Bellow we calculate the criticality equation (Euler-Lagrange equation) of this situation.

1.1.4. *Theorem.* The variation (differential) of the action $S[x] = \ \int_\Sigma d\tau\ L(x(\tau), \dot{x}(\tau), \tau)$, in the direction of $u \in T_x(C^\infty(\Sigma, M)) = \ C^\infty(\Sigma, M))$, is

$$\frac{\delta S}{\delta x} u = \sum_j\ \Big[\ \Big(\prod_{k \ne j} \int_{I_k} d\tau^k\Big)\ L_{\frac{\partial x}{\partial \tau^j}}(x, \dot{x}, \tau)\ u(\tau)\ \Big]_{\tau_j = a_j}^{\tau_j = b_j}\ +\ \int_\Sigma d\tau\ [L_x(x, \dot{x}, \tau) - \frac{d}{d\tau} L_{\dot{x}}(x, \dot{x}, \tau)] \cdot u$$

$$= \int_{\partial\Sigma} d\sigma\ L_{\frac{\partial x}{\partial n}}(x(\sigma), \dot{x}(\sigma), \sigma) \cdot u(\sigma)\ \ +\ \ \int_\Sigma d\tau\ [L_x - \ \frac{d}{d\tau} L_{\dot{x}}](x(\tau), \dot{x}(\tau), \tau)] \cdot u(\tau)$$

for the normal vector n to $\partial\Sigma$.

(b) The EL-equations (criticality equations) are

$$L_x \ =\ \frac{d}{d\tau}(L_{\frac{dx}{d\tau}}) \quad on\ \Sigma,\ and \quad L_{\frac{\partial x}{\partial n}} = 0 \quad on\ \partial\Sigma.$$

(c) The pointwise variation at $t \in \Sigma$ is the integral kernel for the variation operator $\frac{\delta S}{\delta x}$

$$\frac{\delta S}{\delta x(t)} = [L_x - \frac{d}{d\tau} L_{\dot{x}}](x(t), \dot{x}(t), t) \quad + \quad \delta_{t\in\partial\Sigma} \cdot L_{\frac{\partial x}{\partial n}}(x(t), \dot{x}(t), t), \qquad t \in \Sigma.$$

Proof. (a) The variation $\frac{\delta S}{\delta x}$ of S at x and with respect to u, is the differential

$$\frac{\delta S}{\delta x} u = (d_x S)u \stackrel{\text{def}}{=} \frac{d}{d\varepsilon}|_{\varepsilon=0}\, S(x + \varepsilon u) = \int_I d\tau\, \frac{d}{d\varepsilon}|_{\varepsilon=0}\, L(x + u, \dot{x} + \dot{u}, \tau).$$

By differentiating under the integral and using the chain rule this becomes

$$= \iint d\tau\; L_x(x, \dot{x}, \tau)u + L_{\dot{x}}(x, \dot{x}, \tau)\dot{u}.$$

Since $\dot{x}$ is a vector $\frac{\partial x}{\partial \tau^j}$, this is

$$= \iint d\tau^1 \cdots d\tau^p \quad L_x \cdot u^\mu + \; L_{\frac{\partial x}{\partial \tau^j}} \cdot \frac{\partial u}{\partial \tau^j}.$$

It remains to use integration by parts for the "dynamic" ingredients $\frac{\partial u^\mu}{\partial \tau^j}$ (i.e., the Stokes theorem)

$$= \int_\Sigma d\tau\, [L_x - \frac{\partial}{\partial \tau^j} L_{\frac{\partial x}{\partial \tau^j}}] \cdot u \quad + \quad \sum_j \, (\prod_{k\neq j} \int_{I_k} d\tau^k)\; [L_{\frac{\partial x}{\partial \tau^j}}(x, \dot{x}, \tau)\; u(\tau)]_{\tau_j = a_j}^{\tau_j = b_j}.$$

(b) The equation on Σ is obtained by using all u's supported in the interior of Σ and the continuity at the boundary. The equation on $\partial\Sigma$ is then obtained by varying the restriction of u to the boundary. In (c),

$$(d_x S)\, \delta_t(\tau) v \;=\; \int_{\partial\Sigma} d\sigma\, L_{\frac{\partial x}{\partial n}}(x(\sigma), \dot{x}(\sigma), \sigma) \cdot \delta_t(\sigma) v \;+\; \int_\Sigma d\tau\, [L_x - \frac{d}{d\tau} L_{\dot{x}}](x(\tau), \dot{x}(\tau), \tau)] \cdot \delta_t(\tau) v$$

1.1.5. *Remarks.* (a) The variation with respect to x consists of two parts, due to the appearance of x as a *non-dynamical* variable (i.e., the appearance of the field x itself without derivatives), and as a *dynamical* variable (i.e., the appearance of the derivative $\dot{x}$ of the field). The non-dynamical term of the variation has no time derivative, while the dynamical term has time derivative with a minus (since the dynamic variable causes integration by parts).

(b) In general, Σ will be a manifold with boundary and a Riemannian metric that will give the measure used to define the action S. The boundary term in the variation formula will disappear in some cases:

- (i) If Σ is a closed manifold and $L = L(x, \dot{x})$, i.e., $L(x, \dot{x}, \tau)$ does not depend on τ.
- (ii) If we consider maps x with the fixed values on $\partial\Sigma$.

Here, (i) is clear, either by invoking the Stokes theorem or its proof (integral $S[x]$ over Σ can be calculated as a sum of contributions from pieces Σ_k on which one has local coordinates. If we calculate the contribution to $(d_x S)u$ from all pieces, the boundary contributions will cancel. The second case is also obvious since $u = 0$ on $\partial\Sigma$.

(c) We can think of $\dot{x} = \frac{dx}{d\tau}$ as a vector $\frac{\partial x}{\partial \tau^j}$ or a matrix $\frac{\partial x^\mu}{\partial \tau^j}$. Then the criticality equations can be written as

$$L_x = \frac{\partial}{\partial \tau^j} L_{\frac{\partial x}{\partial \tau^j}} \quad L_{x^\mu} = \frac{\partial}{\partial \tau^j} L_{\frac{\partial x^\mu}{\partial \tau^j}}, \quad i.e.,,$$

the variation with respect to the value of x at τ is

$$\frac{\delta S}{\delta x(\tau)} \stackrel{\text{def}}{=} (d_x S)\ \delta_\tau = [L_x - \frac{\partial}{\partial \tau^j} L_{\frac{\partial x}{\partial \tau^j}}](x(\tau), \dot{x}(\tau), \tau),$$

and if we vary only the μ coordinate of x at τ

$$\frac{\delta S}{\delta x^\mu(\tau)} \stackrel{\text{def}}{=} (d_x S)\ \delta_\tau = [L_{x^\mu} - \frac{\partial}{\partial \tau^j} L_{\frac{\partial x^\mu}{\partial \tau^j}}](x(\tau), \dot{x}(\tau), \tau).$$

1.1.6. *The case* $S(x) = \int_{t_1}^{t_2} L(x(t))\ dt$.

1.2. **Borcherds Formalism.** Higher derivatives.

The difference between Lagrangians and EL-equations is that a Lagrangian is to be integrated, so it is rather a measure then a function. (?).

2. Perturbative expansion of Feynman integrals by Feynman graphs

This is one basic idea in quantum field theory.

Part IV. Free strings in a Minkowski space: actions and classical solutions

0. Intro

We consider the area action for free strings as an obvious generalization of the length action for the worldline, Its classical solutions are the minimal submanifolds. It is geometrically reasonable but technically impractical since the square root makes it complicated for quantization.

1. Kinetic energy for a non-relativistic strings propagating on a cylinder

[22] Here, tube has canonical coordinates $\tau^0 =$ (time) and $\tau^1 = \theta$ (angle). They give local charts and a global metric

$$g = \ g_R \stackrel{\text{def}}{=} (c\tau^0)^2 + (R\tau^1)^2$$

which is lifted from $\mathbb{R}^2$. Here, R is the string radius (?).

In the non-relativistic setting, time and space are sharply separated. This product structure forces the worldsheet to be topologically (and metrically?) a cylinder.

2. Area action for free strings ("Nambu-Goto action")

We list here the area action for free strings as an obvious generalization of the length action for the worldline, and we notice its mathematical significance. It is geometrically reasonable but technically impractical, the square root makes it complicated for quantization.

The *volume action* for maps $x : \Sigma \to M$ from a compact oriented manifold with boundary Σ to a manifold M with a metric G, is

$$S[x] \stackrel{\text{def}}{=} \ -T \int_{x(\Sigma)} dV.$$

Here, dV is the volume of a piece of $x(\Sigma)$, calculated in M.

The critical points of this action are called the minimal submanifolds of M (of shape Σ), say geodesics, minimal surfaces etc.

2.0.1. *Remarks.* (1) Actually, we need to restrict to the maps x such that x^*G (or $-x^*G$), is a positive definite metric on Σ. (2) Moreover, if x is not defined on $\partial\Sigma$ we need to ask that the integral converges.

[22] TO BE DONE

2.0.2. *String tension constant T.* It has to appear to make the action dimensionless (for instance, we need to exponentiate the action for the path integrals). For that,

$$\dim(T) = \ volume^{-1} = \ length^{-\dim(\Sigma)}.$$

2.0.3. *Question.* Are the energy and volume actions always physically equivalent ?

The question is not precise. Say, the energy actions seem to require an additional field: the metric on the source, so that we can measure the speed, i.e., the size of velocity.

The answer seems to be positive for particles and strings!

2.0.4. *Formulas in coordinates.* We will denote local coordinates on Σ by $\tau^0, ..., \tau^{\dim(\Sigma)}$. Let $\partial_i \stackrel{\text{def}}{=} \partial_{\tau^i}$. Since x is a vector $x = (x^\mu)$, its partials are $\partial_i x \stackrel{\text{def}}{=} dx(\partial_i) \ = \ \partial_i x^\mu \cdot \frac{\partial}{\partial X^\mu}$.

Lemma. (a) The pull-back metric $g = \ x^*G$ on Σ is

$$g_{ij} = \ \partial_i x \underset{G}{\cdot} \partial_j x = \ G_{\mu\nu} \cdot \partial_i x^\mu \cdot \partial_j x^\nu.$$

(b) The induced volume on Σ is

$$dV = \ d\tau^0 d\tau^1 \cdots d\tau^{\dim(\Sigma)} \cdot \sqrt{-det(g_{ij})}.$$

(Actually, the sign under the square root appears if we are dealing with a metric G on M that has Minkowski nature.)

(c) The action is

$$S[x] \stackrel{\text{def}}{=} \ -T \int_\Sigma d\tau^0 d\tau^1 \cdots d\tau^{\dim(\Sigma)} \sqrt{-det(g_{ij})} = \ -T \int_\Sigma d\tau^0 d\tau^1 \cdots d\tau^{\dim(\Sigma)} \ L(\dot{x});$$

for the Lagrangian with one, purely dynamical, variable x

$$L(\dot{x}) = \ -T \cdot \sqrt{-det(g_{ij})} = \ -T \cdot \sqrt{-\det(G_{\mu\nu} \cdot \partial_i x^\mu \partial_j x^\mu)}.$$

Proof. (a) $g_{ij} = \ (x^*G)_{ij} \stackrel{\text{def}}{=} (x^*G)(\partial_i, \partial_j)$ equals

$$G(dx\,\partial_i, dx\,\partial_j) = \ (dx\,\partial_i) \underset{G}{\cdot} (dx\,\partial_j) = \ \partial_i x \underset{G}{\cdot} \partial_j x = \ \partial_i x^\mu \frac{\partial}{\partial X^\mu} \underset{G}{\cdot} \partial_j x^\nu \frac{\partial}{\partial X^\nu} = \ G_{\mu\nu} \cdot \partial_i x^\mu \cdot \partial_j x^\nu.$$

(b) The change of variables in the area integral, corresponding to the parameterization x, gives $dV = \ d\tau^0 d\tau^1 \cdots d\tau^{\dim(\Sigma)} \cdot \sqrt{-det(g_{ij})}$. Actually, the sign under the square root appears if we are dealing with a metric G on M that has Minkowski nature.

2.0.5. *Minimal submanifolds.* We say that x or $x(\Sigma)$is a minimal submanifold of M if x is a critical point for the area action.

We've got nothing to say about this except that it is a complicated subject:

Corollary. EL-equation is

$$0 = \frac{\delta S}{\delta x^\mu} = \sum_j \partial_j \, L_{\partial_j \partial x^\mu} = \sum_j \frac{\partial}{\partial_{\tau_j}} \, L_{\frac{\partial x^\mu}{\partial \tau_j}}.$$

2.0.6. *The case when Σ is a surface.* Consider an oriented surface with boundary Σ (a smooth 2dimensional compact manifold with boundary). The strings with evolution of type Σ are the maps $x : \Sigma \to M$.

The *area action* of the map x will be the area of the image $x(\Sigma) \subseteq M$ with respect to the metric G on M,

$$S[x] \stackrel{\text{def}}{=} -T \int_{x(\Sigma)} dA.$$

The local coordinates a surface on Σ are τ^0, τ^1; and

- $-det(g_{ij}) = (g_{01})^2 - g_{00} \cdot g_{11} = (\partial_0 x \underset{G}{\cdot} \partial_1 x)^2 - (\partial_0 x)^2_G \, (\partial_1 x)^2_G$,
- the action is $S[x] \stackrel{\text{def}}{=} -T \int_\Sigma d\tau^0 d\tau^1 \cdot \sqrt{-det(g_{ij})} = -T \int_\Sigma d\tau^0 d\tau^1 \cdot L(\dot{x})$,
- and the Lagrangian is $L(\dot{x}) = -T \sqrt{(\partial_0 x \underset{G}{\cdot} \partial_1 x)^2 - (\partial_0 x)^2_G \, (\partial_1 x)^2_G}$.

3. Polyakov action for free strings (kinetic action with added worldsheet metric)

In this section we calculate the criticality equations for the Polyakov action for strings in a manifold with a "constant" metric. For later use we restate the results in special coordinates ("conformally flat coordinates" and "conformal gauge").

3.0.7. *Polyakov action.* By the kinetic energy action of a map $\Sigma \xrightarrow{x} M$ we will mean the integral over Σ of the velocity squared. This requires metrics G, g on both M and Σ in order to measure the size of velocity:

$$S[x,g] \stackrel{\text{def}}{=} T \cdot \int_{x(\Sigma)} dV \ \frac{-||dx||^2_{G,g}}{2} \ .$$

More precisely, we will see that this works precisely for strings, i.e., when $\dim(\Sigma) = 2$. For other branes one needs additional (massive?) terms in the action.

3.0.8. *Comparison with the volume action and criticality in g.* Here, we need one additional field, the worldsheet metric g. However, we will find that this new degree of freedom disappears classically, i.e., at the criticality of the action $S[x,g]$ in g. This criticality equation expresses g in terms of x, and when we substitute this formula for g into $S[x,g]$, the $S[x,g]$ reduces to the area action $S_{Area}[x]$.

3.0.9. *Criticality in x.* The solutions of the criticality equation in x are classical mathematical objects, the harmonic maps.

3.0.10. *Massless strings.* This action is quadratic in the derivative dx and it allows for massless strings. This is important since we are most interested in the massless case. The appearance of massless objects (gravitons) is one of the key attractions of the string theory.

3.0.11. *Classical solutions for the Polyakov action.* The main result of this section is summarized in the following theorem which is proved in lemmas 3.3.1 and 3.5.3 bellow.

Theorem. The classical solutions for the Polyakov action are the pairs (x, g) with

- g conformal to the pull-back metric x^*G and
- x a harmonic map for the metrics G and x^*G.

3.0.12. *Symmetries.* Besides the usual ones (automorphisms of Σ and (M, G)), Polyakov action for surfaces (only!) has additional conformal symmetry.

3.0.13. *Stress tensor* $\boldsymbol{T}$. Here it appears just as a name for the (normalized) variation in g^{-1}. So,

$$\boldsymbol{T} = 0 \text{ is the criticality equation for the non-dynamical variable.}$$

In the **remainder**, i.e., starting from the subsection E.1, we just restate the above computations in terms of special coordinates on Σ.

3.0.14. *Conformally flat complex coordinates.* To a metric g on Σ one can associate a complex structure on Σ and then it is convenient to calculate in terms of a local holomorphic coordinate z. We say that such complex coordinates z is conformally flat for g.

3.0.15. *Conformal gauge.* We notice that in the local computations it is legitimate to assume that we are in the *Conformal gauge*, i.e., that $g_{ij} = \delta_{ij}$.

3.1. **Polyakov action.** Historically, it was discovered by two groups , . and , ; and popularized by Polyakov.

The setting consists of a metric spacetime (M, G) and the worldsheet manifold with boundary Σ. There are two fields, maps $x : \Sigma \to M$ and worldsheet metrics g. Let $s = \dim(\Sigma)$.

The action of a map $\Sigma \xrightarrow{x} M$ and the worldsheet metric g is the integral over Σ of the velocity squared

$$S[x, g] \stackrel{\text{def}}{=} T \cdot \int_{x(\Sigma)} dV \ \frac{-||dx||^2_{G,g}}{2} \ = \ -\frac{T}{2} \int_{x(\Sigma)} dV \ dx \underset{G,g}{\cdot} dx = \ -\frac{T}{2} \int_{\Sigma} d\tau^0 \cdots d\tau^s \, L(dx, g),$$

for the Lagrangian

$$L(dx, g) \stackrel{\text{def}}{=} \sqrt{-\det(g_{pq})} \cdot \partial_{\tau_i} x \underset{G}{\cdot} \partial_{\tau_j} x \cdot g^{ij} = \ \sqrt{-\det(g_{pq})} \cdot \partial_i x^\mu \cdot \partial_j x^\nu \cdot G_{\mu\nu} \cdot g^{ij}.$$

3.1.1. *Remarks.*

(1) We will do computations in general, but we will see that the above action only works when Σ is a surface, i.e., for strings.

For other branes it may be that one needs an additional massive term, We know that this is what happens when $\dim(\Sigma) = 1$, i.e., for particles (3.3). Though massless particle formally makes sense, there are no classical solutions (3.3).

(2) At each point $a \in \Sigma$, the square size $-(d_a x)^2_{G,g} \stackrel{\text{def}}{=} -d_a x \underset{G,g}{\cdot} d_a x$, of the differential $d_a x \in \mathrm{Hom}_{\mathbb{R}}[T_a\Sigma, T_{x(a)}M]$, is measured with respect to the bilinear form $g_a^* \otimes G_{x(a)}$ on $\mathrm{Hom}_{\mathbb{R}}[T_a\Sigma, T_{x(a)}M]$ (see lemma C.1.2.e).

(3) The *brane tension constant* T appears to make the action dimensionless (for instance, we need to exponentiate the action for the path integrals).

(4) The minus sign appears because $||dx||^2_{G,g} < 0$ for Lorentzian G.

3.1.2. *The use of local coordinates.* We symbolically write the integral in local coordinates, as if there exist global coordinates. The meaning is that in reality the integral over Σ is a sum of contributions calculated in local coordinates. The use of local coordinates is convenient since our calculus of variations formulas have been deduced for the action written in local coordinates.

3.1.3. *Why do we need the metric g ?* (1) If g is conformal to the pull-back metric $g = \phi \cdot x^*G$, then

$$||d_a x||^2_{G,\phi\cdot x^*G} = \phi(a)^{-1} \cdot ||d_a x||^2_{G,x^*G} = \frac{\dim(\Sigma)}{\phi(a)} .$$

For the last equality see lemma C.1.2.e, or directly, observe that if x is regular at a, then $d_a x : T_a\Sigma \to T_{x(a)}M$, viewed as a map to its image, is an isomorphism of vector spaces with inner products. so it is essentially the identity on $T_a\Sigma$.)

(2) In particular, the obvious choice $g = x^*G$ is not good for measuring speed since $||d_a x||^2_{G,x^*G} = \dim(\Sigma)$.

(3) However, the question of

For a given Σ and (M, G), find maps $\Sigma \xrightarrow{x} (M, G)$, harmonic for G and x^*G.

is still interesting. It is quite different from the standard question on harmonic maps:

For given (Σ, g) and (M, G) find all harmonic maps $x : \Sigma \to M$.

3.1.4. *The signature of the metric g.* We will use

- *Lorentzian g for local reasoning.* In particular, to see that Polyakov action reduces to the area action at the criticality in g.
- *Euclidean g for global reasoning.* Actually, on a topologically non-trivial (i.e., $g > 0$) surface Σ there are no non-singular Lorentzian metrics (it would give a decomposition of $T\Sigma$ into two isotropic line bundles).

So we will eventually be interested in Riemannian g, though Lorentzian metric is physically natural (remember that we used the Lorentzian metric x^*G for the area action). The justification of the transition is as usual through the Wick rotation (1.3).

3.2. **Symmetries of the action.** We say that two metrics g', g'' are conformally equivalent if $g'' = \phi \cdot g'$ for a positive function ϕ. Besides the usual symmetries, in the case of strings, i.e., when $\Sigma$is a surface, the Polyakov action also has conformal invariance:

Lemma. The symmetries of Polyakov action $S[x,g]$ contain

(1) *Spacetime symmetries.* The Poincare group $\mathcal{P}_M$ consists of diffeomorphism P of the spacetime M that preserve the metric G. (So, $P \in \mathcal{P}_M$ moves x to $P \circ x$ and it fixes g).
(2) *Reparameterizations.* The group $Diff_0(\Sigma)$, of orientation preserving reparameterizations of Σ. ($\sigma \in Diff_0(\Sigma)$ moves (x,g) to $(x \circ \sigma^{-1}, \sigma^* g)$.)
(3) *Rescalings of g.* If $\dim(\Sigma) = 2$, one also has the group $C^\infty(\Sigma, \mathbb{R}_{>0}$ of conformal symmetries. A positive function $\phi \in C^\infty(\Sigma, \mathbb{R}_{>0}$ moves (x,g) to $(x, \phi \cdot g)$. This preserves both the action and the Lagrangian.

Proof. (c) Since the volume is the square root of the determinant of the metric g_{ij}

$$(dV)_{\phi \cdot g} = \phi^{\dim(\Sigma)/2} \cdot (dV)_g, \quad \text{while} \quad (\phi g)^{ij} = \phi^{-1} g^{ij} \text{ gives } (dx)^2_{G,\phi \cdot g} = \phi^{-1} (dx)^2_{G,g}.$$

So, $L(\dot{x}, \phi \cdot g) = \phi^{\dim(\Sigma)/2-1} \cdot L(\dot{x}, g)$, and one needs $\dim(\Sigma) = 2$ for invariance.

3.3. **Variation with respect to the inverse metric.**

3.3.1. *Lemma.* (a) For the non-dynamical variable g^{-1},

$$\frac{\delta S}{\delta g^{ij}} = -\frac{T}{2} \cdot (-\det(g_{rs}))^{\frac{1}{2}} \cdot [\partial_{\tau_i} x \underset{G}{\cdot} \partial_{\tau_j} x - \frac{1}{2} g_{ij} \cdot (d_a x)^2_{G,g}].$$

(b) *Criticality in g^{-1}.* If $\dim(\Sigma) = 2$ then EL-equation says that g is conformal to the pull-back of $-G$, i.e., $g = \phi \cdot x^* G$ for arbitrary ϕ. (If $\dim(\Sigma) \neq 2$, then the criticality in g^{-1} has no solutions at all!)

(c) If Σ is a surface, at the criticality for g Polyakov action reduces to the area action:

$$S_{Poly}[x, \phi \cdot (-x^* G)] = S_{Poly}[x, -x^* G] = S_{Area}[x].$$

Proof. (a) Since g^{-1} is a non-dynamical variable $\frac{\delta S}{\delta g^{ij}}$ equals

$$L_{g^{ij}} = -\frac{T}{2} \frac{\partial}{\partial g^{ij}} [(-\det(g^{rs}))^{-\frac{1}{2}} \partial_{\tau_p} x \underset{G}{\cdot} \partial_{\tau_q} x \cdot g^{pq}]$$

$$= -\frac{T}{2} [-\frac{1}{2} (-\det(g^{rs}))^{-\frac{3}{2}} \cdot \frac{\partial}{\partial g^{ij}} (-\det(g^{rs})) \cdot (d_a x)^2_{G,g} + (-\det(g^{rs}))^{-\frac{1}{2}} \partial_{\tau_i} x \underset{G}{\cdot} \partial_{\tau_j} x]$$

With the formula for the derivative of the determinant (lemma C.1.3.b), this is

$$= -\frac{T(-\det(g_{rs}))^{\frac{1}{2}}}{2} [\partial_{\tau_i} x \underset{G}{\cdot} \partial_{\tau_j} x - \frac{1}{2} (-\det(g^{rs}))^{-1} \cdot (-\det(g^{rs})) g_{ij} \cdot (d_a x)^2_{G,g}]$$

$$= -\frac{T(-\det(g_{rs}))^{\frac{1}{2}}}{2} [\partial_{\tau_i} x \underset{G}{\cdot} \partial_{\tau_j} x - \frac{1}{2} g_{ij} \cdot (d_a x)^2_{G,g}].$$

(b) Criticality equation $\frac{\delta S}{\delta g^{ij}} = 0$ says that $\partial_{\tau_i} x \cdot_G \partial_{\tau_j} x = \frac{(d_a x)^2_{G,g}}{2} g_{ij}$, i.e.,

$$g \;=\; \frac{2}{-||d_a x||^2_{G,g}} \cdot (-x^* G).$$

So g is conformal to the pull back metric $g \;=\; \phi \cdot\; x^* G$. Now, by 3.1.3, $||d_a x||^2_{G,g} = ||d_a x||^2_{G,\phi \cdot x^* G} \;=\; \phi(a)^{-1} \cdot \dim(\Sigma)$, hence $\phi(a) \;=\; \frac{2}{||d_a x||^2_{G,g}} \;=\; \frac{2\phi(a)}{\dim(\Sigma)}$. This shows that ϕ is arbitrary when $\dim(\Sigma) = 2$, and that there are no solutions at all when $\dim(\Sigma) \neq 2$.

(c) At the criticality, $g = \phi \cdot (-x^* G)$ for some ϕ. If Σ is a surface, factor ϕ is not a problem since Polyakov action is conformally invariant. So,

$$S[x, -\phi \cdot x^* G] = \; S[x, -x^* G] \;=\; -\frac{T}{2} \int_{x(\Sigma)} dV \;\; ||dx||^2_{G,-x^* G}.$$

However, for $g = -x^* G$ one has $||dx||^2_{G,-x^* G} \;=\; -\dim(\Sigma) \;=\; -2$ (see 3.1.3). Therefore, $S[x, -x^* G] \;=\; -\frac{T \dim(\Sigma)}{2} \;\; \int_{x(\Sigma)} \; dV \;=\; S_{Action}[x]$.

3.4. **Stress tensor $\boldsymbol{T}$.** This is the normalized variation of the action with respect to the non-dynamical variable g^{-1}

$$\boldsymbol{\mathcal{G}}\; \boldsymbol{T}(x, g) \stackrel{\text{def}}{=} \; -\frac{2}{T \cdot \sqrt{-\det(g_{rs})}} \cdot \frac{\delta S}{\delta g^{-1}}.$$

Lemma. (a) In coordinates

$$\boldsymbol{T}_{ij}(x, g) \stackrel{\text{def}}{=} \frac{-2}{T \cdot \sqrt{-\det(g_{rs})}} \frac{\delta S}{\delta g^{ij}} = \; \partial_{\tau_i} x \cdot_G \partial_{\tau_j} x - \; \frac{1}{2} g_{ij} \cdot (d_a x)^2_{G,g}.$$

(b) The stress tensor $\boldsymbol{T}(x, g)$ is a *quadratic differential* on Σ, i.e., a family of *symmetric bilinear forms* on tangent spaces. (Just like metric!)

Proof. (a) has been calculated above. (b) At a point a in Σ,the stress tensor $\boldsymbol{T}_a[x, g]$ is related to changing $g^{-1} = (g^{rs})$ at the point a, so it is a linear functional on the vector space of possible values of g^{-1} at a. The value of g^{-1} at a is a symmetric bilinear form $(g^{-1})_a$ on the vector space $T^*_a \Sigma$. However, $(g^{-1})_a$ lies in the symmetric part of the dual of $\mathrm{Hom}[T^*_a \Sigma \otimes T^*_a \Sigma, \mathbb{C}]$, i.e., in $S^2(T_a \Sigma)$.

3.5. **Variation with respect to the map x (harmonic maps).**

3.5.1. *Criticality in x (harmonic maps).* The maps that satisfy the criticality condition in variable x are called the (G, g)-*harmonic maps*.

We will only consider the case when M is a vector space and the metric G is constant. In the case when Σ is a surface, the criticality equations simplify in special coordinates – one can choose local coordinates in which metric g is conformal to the standard metric on $\mathbb{R}^2$. Instead of doing this for Lorentzian metrics g that we use here, we will do this

later (see 1.4), in the setting of Riemannian metrics g that we use in order to quantize the strings.

3.5.2. *Laplace operator.* We will use the Laplace operator for the metric g on Σ,

$$\nabla_g^2 \stackrel{\text{def}}{=} g^{ij}\partial_{\tau_i}\partial_{\tau_j} = g^{ij}\partial_i\partial_j.$$

3.5.3. *Lemma.* For the purely dynamical variable x

$$\frac{\delta S}{\delta x^\mu} = \sqrt{-\det(g_{rs})}\cdot\nabla_g^2(x_\mu) + \partial_{\tau_i}x_\mu\cdot\partial_{\tau_j}\,[g^{ij}\sqrt{-\det(g_{rs})}]$$
$$= \sqrt{-\det(g_{rs})}\cdot\nabla_g^2(x_\mu) + \partial_i x_\mu\cdot\partial_j\,[g^{ij}\sqrt{-\det(g_{rs})}].$$

Proof. Since x appears in the Lagrangian through dx,

$$\frac{\delta S}{\delta x^\mu} = -\sum_j \frac{\partial}{\partial\tau_j}\,L_{\frac{\partial x^\mu}{\partial\tau_j}},$$

so we first calculate

$$\frac{\partial}{\partial\,(\frac{\partial x^\mu}{\partial\tau_j})}\,L = \frac{\partial}{\partial\,(\frac{\partial x^\mu}{\partial\tau_j})}[-\frac{T\sqrt{-\det(g_{rs})}}{2}\cdot\partial_p x^\phi\cdot\partial_q x^\psi\cdot G_{\phi\psi}\cdot g^{pq}]$$

$$= -\frac{T\sqrt{-\det(g_{rs})}}{2}\cdot(\partial_q x^\psi\cdot G_{\mu\psi}\cdot g^{jq} + \partial_p x^\phi\cdot G_{\phi\mu}\cdot g^{pj}) = -\frac{T\sqrt{-\det(g_{rs})}}{2}\cdot 2\partial_p x^\nu\cdot G_{\mu\nu}\cdot g^{pj}.$$

Because the metric G is constant, $\partial_p x^\nu\cdot G_{\mu\nu} = \partial_p(x^\nu\cdot G_{\mu\nu}) = \partial_p x_\mu$, so

$$= -\frac{T\sqrt{-\det(g_{rs})}}{2}\cdot 2\partial_p x_\mu\cdot g^{pj} = -T\sqrt{-\det(g_{rs})}\cdot g^{jp}\cdot\partial_{\tau_p}x_\mu.$$

Then,

$$\frac{\delta S}{\delta x^\mu} = -\frac{\partial}{\partial\tau_j}\,[-T\sqrt{-\det(g_{rs})}\cdot g^{jp}\cdot\partial_p x_\mu] = T\cdot\partial_j[\partial_p x_\mu\cdot g^{jp}\cdot\sqrt{-\det(g_{rs})}]$$

and

$$\frac{1}{T}\cdot\frac{\delta S}{\delta x^\mu} = \partial_j\partial_p x_\mu\cdot g^{jp}\cdot\sqrt{-\det(g_{rs})} + \partial_p x_\mu\cdot\partial_j[g^{jp}\sqrt{-\det(g_{rs})}].$$

So it remains to recognize $g^{jp}\partial_j\partial_p$ as the Laplace operator ∇_g^2 for the metric g on Σ.

3.6. **Relation between metrics and complex structures on surfaces.** Let us recall the relation between metrics and complex structures on surfaces from the section IX.E.

On an oriented surface (Σ, or) a complex structure J is the same as a conformal class of Riemannian metrics g. We will use expression *conformally flat* to invoke the above relation of of a metric g and a complex structure J (or any local holomorphic coordinate z). In this way some metric questions become questions in holomorphic geometry (i.e., some computations for a given metric g can be done using the associated complex structure J).

We will compute the stress tensor $\boldsymbol{T} = \boldsymbol{T}[x, g]$ (i.e., criticality in g), in terms of a holomorphic coordinate z conformally flat to g. In particular, at the criticality in x stress tensor $\boldsymbol{T}$ will reduce to one holomorphic function $\boldsymbol{T}(z) \stackrel{\text{def}}{=} \boldsymbol{T}(\partial, \partial)$.

3.6.1. *Computations in conformally flat coordinates.* Consider a Riemannian metric g on an oriented surface (Σ, or) and a conformally flat local holomorphic coordinate z. The corresponding real coordinates are $\tau^1 = Re(z), \tau^2 = Im(z)$. with real . In these coordinates g has a very simple form (lemma IX.E.3.1):

(1) $g_{ij} = \delta_{ij}\omega, \quad g^{ij} = \delta_{ij}\omega^{-1}$ for $\omega = g_{ii} \in C^\infty(\Sigma, \mathbb{R}_{>0})$.

(2) $\begin{pmatrix} g_{zz} & g_{z\bar z} \\ g_{\bar z z} & g_{\bar z \bar z} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{pmatrix}$ and $\begin{pmatrix} g^{zz} & g^{z\bar z} \\ g^{\bar z z} & g^{\bar z \bar z} \end{pmatrix} = \begin{pmatrix} 0 & 2/\omega \\ 2/\omega & 0 \end{pmatrix}, \quad \omega = 2g_{z\bar z} = 2g(\partial, \bar\partial)$.

(3) The g-Laplacian is $\nabla_g^2 \stackrel{\text{def}}{=} g^{ij}\partial_i\partial_j = \frac{1}{\omega}\partial\bar\partial$.

(4) The velocity of a map $(\Sigma, g) \stackrel{x}{\longrightarrow} (M, G)$ is $(dx)^2_{G,g} = \frac{4}{\omega}\, \partial x \underset{G}{\cdot} \bar\partial x$.

3.6.2. *Stress tensor $\boldsymbol{T}$ in conformally flat coordinates.* The definition $\boldsymbol{T}(x, g) \stackrel{\text{def}}{=} -\frac{2}{T\cdot\sqrt{\det(g_{rs})}}\cdot\frac{\delta S}{\delta g^{-1}}$, translates in any local complex coordinate z to

$$\boldsymbol{T}_{zz} \stackrel{\text{def}}{=} -\frac{2}{T\cdot\sqrt{\det(g_{rs})}}\cdot\frac{\delta S}{\delta g^{zz}}, \quad etc.$$

Lemma. (a) In any local complex coordinate on Σ, the stress tensor is given by

$$\boldsymbol{T}_{zz} = \partial x \underset{G}{\cdot} \partial x - \frac{1}{2} g_{zz}\, (dx)^2_{G,g} = \overline{\boldsymbol{T}_{\bar z \bar z}} \quad \text{and} \quad \boldsymbol{T}_{z\bar z} = \partial x \underset{G}{\cdot} \bar\partial x - \frac{1}{2} g_{z\bar z}\, (dx)^2_{G,g}.$$

(b) In conformally flat coordinates $\boldsymbol{T}(z) \stackrel{\text{def}}{=} \boldsymbol{T}_{zz}$ is holomorphic, $\widetilde{\boldsymbol{T}}(\bar z) \stackrel{\text{def}}{=} \boldsymbol{T}_{\bar z \bar z}$ is antiholomorphic, and $\boldsymbol{T}_{z\bar z} = 0$. Actually,

$$\boldsymbol{T}(z) \stackrel{\text{def}}{=} \boldsymbol{T}_{zz} = \partial x \underset{G}{\cdot} \partial x, \qquad \widetilde{\boldsymbol{T}}(\bar z) \stackrel{\text{def}}{=} \boldsymbol{T}_{\bar z \bar z} = \bar\partial x \underset{G}{\cdot} \bar\partial x.$$

(c) Moreover, if (x, g) is a critical point of the Polyakov action $S[x, g]$ with respect to x (i.e., x is (g, G)-harmonic), then $\boldsymbol{T}(z) = \boldsymbol{T}(x, g)(z) = \boldsymbol{T}_{zz}(x, g)$ is a holomorphic function of complex coordinate z (which is conformally flat for the metric g).

Proof. (a) $\boldsymbol{T}$ is a quadratic differential, so it transforms the same as a metric, i.e., its ij-coordinates transform to the $z, \bar z$ coordinates the same as $\partial_i \otimes \partial_j$ transforms into $\partial \otimes \partial$ etc. So, the formula above is the same as the basic formulas (lemma 3.4)

$$\boldsymbol{T}_{ij}(x, g) = \partial_i x \underset{G}{\cdot} \partial_j x - \frac{1}{2} g_{ij} \cdot (d_a x)^2_{G,g}.$$

(In general, $g_{\bar z \bar z} = \overline{g_{zz}}$.)

(b) If our complex coordinate z is conformally flat for g then $g_{zz} = 0$ (lemma IX.E.3.1), hence

$$\boldsymbol{T}(z) \stackrel{\text{def}}{=} \boldsymbol{T}_{zz} = \partial x \underset{G}{\cdot} \partial x - \frac{1}{2} g_{zz}\ (dx)^2_{G,g} = \partial x \underset{G}{\cdot} \partial x \quad \text{and} \quad \widetilde{\boldsymbol{T}}(z) \stackrel{\text{def}}{=} \boldsymbol{T}_{\bar{z}\bar{z}} = \overline{\boldsymbol{T}_{zz}} = \overline{\boldsymbol{T}(z)}.$$

Similarly, $\boldsymbol{T}_{z\bar{z}} = \partial x \underset{G}{\cdot} \bar{\partial} x - \frac{1}{2} g_{z\bar{z}}\ (dx)^2_{G,g}$ vanishes since lemma IX.E.3.1 (parts b and d), says that

$$g_{z\bar{z}} \cdot (dx)^2_{G,g} = \frac{\omega}{2} \cdot \frac{4}{\omega} \partial x \underset{G}{\cdot} \bar{\partial} x = 2 \partial x \underset{G}{\cdot} \bar{\partial} x.$$

(c) The criticality in x means that x is harmonic for g and G, hence for g_{st} and G, and this implies that ∂x is holomorphic. So, $\boldsymbol{T}(z) = \boldsymbol{T}(x, g)(z) = \partial x \underset{G}{\cdot} \partial x$ is holomorphic.

3.7. **Conformal gauge.** By a *gauge* we mean the passage to a subspace $\mathcal{F}_{gauge}$ of the the space $\mathcal{F}$ of all fields, which generates all of $\mathcal{F}$ under the action of the group of symmetries of the action (the *gauge group*).

3.7.1. *Conformal gauge.* In our cause the space of fields is a product $\mathcal{F} = Map(\Sigma, M) \times Metrics(\Sigma)$.

Here, we will assume that the metric g is the standard or *fiducial* metric $g_{st} = \widehat{g}$ with $\widehat{g}_{ij} = \delta_{ij}$ for the global coordinates τ^1, τ^2 on Σ. In other words we pass to the subspace $\mathcal{F}_{gauge} \stackrel{\text{def}}{=} Map(\Sigma, M) \times \{g_{st}\}$ of $\mathcal{F}$.

The lemma IX.E.3.2 says that for any metric g on a surface one can locally choose coordinates so that in these coordinates g is the *fiducial* metric g_{st}. So we are in some (at the moment hazy) sense, looking at the "Σ-local properties of the path integral". We can also restate the lemma as: locally the choice $g = g_{st}$ is a *gauge* (since the change of coordinates on Σ is the part of the symmetry group of the action). So, locally the *conformal gauge* really is a gauge.[23]

We will calculate in terms of the corresponding complex coordinate $z = \tau^1 + i\tau^2$ which is conformally flat for g.

3.7.2. *EL-equations in the conformal gauge.* Since the criticality equations are local, it is quite legitimate to compute the criticality in x in the conformal gauge $g = g_{st}$. The result is

[23]To use the conformal gauge globally one would need to deal with the remaining global problem of patching the local contributions.

Lemma. In the conformal gauge (i.e., when $g = g_{st}$),

(a) The criticality condition in variable x just means that the component functions x_μ are the harmonic functions in the classical sense.

(b) The criticality condition in g, calculated at $\widehat{g}$ is

$$\partial x \underset{G}{\cdot} \partial x = 0.$$

Proof. (a) Recall that (for Riemannian g)

$$\frac{1}{T}\frac{\delta S}{\delta x^\mu}(x, g) = \nabla_g^2 x_\mu \cdot \sqrt{\det(g_{rs})} + \partial_{\tau_\psi} x_\mu \cdot \partial_{\tau_j} [g^{jp}\sqrt{\det(g_{rs})}].$$

Since $\widehat{g}$ is constant and $\det(\widehat{g}_{rs}) = 1$,

$$\frac{1}{T}\frac{\delta S}{\delta x^\mu}(x, \widehat{g}) = \nabla_{\widehat{g}}^2 x_\mu = (\partial_1{}^2 + \partial_2{}^2) x_\mu = \partial\bar{\partial} x_\mu,$$

with the last line using the complex coordinates.

(b) The criticality condition in g is the vanishing of the stress tensor, and by 3.4(b), this reduces to

$$\partial x \underset{G}{\cdot} \partial x = 0.$$

(Recall that we have extended G to the complexification in a $\mathbb{C}$-bilinear way, hence even if G were positive this would mean that $\partial G = 0$.)

Part V. Free strings in a Minkowski space: quantization

0. Intro

1. Polyakov path integral

In this section we only formulate the meaning of the path integral for the Polyakov action. This involves

- passing from the physically interesting Minkowski formulation to the mathematically more convenient Euclidean formulation by *Wick rotation*.
- Definition of the integral as an equivariant integral for the gauge group $\boldsymbol{\mathcal{G}}$.
- The philosophy of the computation of equivariant integrals in terms of a *gauge*.
- Computations in the *Conformal gauge*.

We will not make any conclusions here from the path integral. So the physics is postponed to the next section in which we study the quantization of a free string in the operator formalism (deformation quantization).

1.0.3. *Path integral.* We are interested in the path integral

$$\int DxDg\ e^{\frac{i}{\hbar}S[x,g]}$$

over Lorentzian metrics g on an oriented surface Σ and maps x from Σ to the Minkowski space M. However, we will define and study it using the its Euclidean version:

- Mathematically, we can define such integrals only by analytic continuation of the corresponding Euclidean path integrals. This procedure is called the *Wick rotation*.
- The Euclidean version is related to algebraic geometry since oriented Riemann surfaces are complex curves.

1.0.4. *Use of the complex structure associated to a metric g.* Our first step is to associate to the field g (a metric on a oriented surface Σ), a complex structure on Σ (E.1). In terms of a local holomorphic coordinate z (such coordinate is said to be *conformally flat for g*, we will recalculate the criticality equations for the Polyakov action. The criticality of x (harmonicity for G, g), turns out to be the standard harmonicity for this complex structure, i.e., the requirement that ∂x is holomorphic. We have formulated the criticality in g as the vanishing of the stress tensor $\boldsymbol{T}[x, g]$. In this setting, at the criticality of x we find that $\boldsymbol{T}[x, g]$ is given by one holomorphic function $\boldsymbol{T}(z)$.

1.0.5. *Computation of a path integral using a gauge.* On the space of fields $\mathcal{F}$ we have the action of the symmetry group of the action, called the *gauge group* $\boldsymbol{\mathcal{G}}$. It consists of reparameterizations and rescalings of the metric: $\boldsymbol{\mathcal{G}} = Diff(\Sigma, or) \times C^\infty(\Sigma, \mathbb{R}_{>0})$.

The meaning of the path integral over $\mathcal{F}$ is really the integral over $\mathcal{F}/\boldsymbol{\mathcal{G}}$. A *gauge* is a subspace $\mathcal{F}_{gauge} \subseteq \mathcal{F}$ of fields such that $\mathcal{G} \cdot \mathcal{F}_{gauge} = \mathcal{F}$. It allows rethinking the integral as an an integral over $\mathcal{F}_{gauge}/\boldsymbol{\mathcal{G}}_{gauge}$ where the *residual gauge group* $\boldsymbol{\mathcal{G}}_{gauge}$ is the stabilizer of $\mathcal{F}_{gauge}$ in $\boldsymbol{\mathcal{G}}$.

1.0.6. *Conformal gauge.* In our case the space of fields $\mathcal{F} = Map(\Sigma, M) \times Metrics(\Sigma)$ is a product of maps and metrics, and the gauge group consists of reparameterizations and rescalings of the metric: $\boldsymbol{\mathcal{G}} = Diff(\Sigma, or) \times C^\infty(\Sigma, \mathbb{R}_{>0})$.

The *conformal gauge* is obtained by fixing the metric g to be the standard metric g_{st}. This in particular fixes a complex structure on Σ. The residual gauge group is then (isomorphic to) the group of conformal automorphisms of Σ.

1.0.7. *The criticality equations in the conformal gauge.* The condition in x is that the component functions x_μ are (ordinary) harmonic functions (for the above complex structure on Σ). The condition in g, calculated at g_{st} is $\partial x \cdot_G \partial x = 0$ (again, ∂ uses the fixed complex structure on Σ).

1.1. **Some comments.** [24]

1.1.1. *Quantum corrections.* At the moment we will be concerned with local computations, so Σ will really be a disc. We will consider the amplitude for a cylinder, and later we add the amplitudes for topologically non-trivial evolutions (quantum corrections or instanton corrections).

1.1.2. *Scattering amplitudes (S-matrix).* One can really calculate the amplitudes only when the sources are at infinity, i.e., $\partial x(\Sigma)$ is on ∂M. However, by conformal transforms a tube going to ∞ becomes a disc without a point. So, the consequence for us is that Σ is not a manifold with boundary, but a compact surface with a few points removed!

An incoming or outgoing string is then represented on an infinite incoming or outgoing tube by a vertex operator.

1.1.3. *Open and closed strings.* Any theory with open strings must contain closed strings since an open string can have an evolution in which ends join. Another way to say this: suppose that an open string splits into two for a moment and then they again join. The evolution surface is an annulus, so if we slice it differently (i.e., measure time in a different way) it can be seen as an evolution of a closed string.

1.1.4. *Free brane.* Really the only example in this course is the free brane, i.e., free particle or free string.

[24] Where should these comments be?

1.2. **Polyakov path integral in Euclidean setting.** We will consider the path integral

$$\int DxDg \quad e^{-\frac{1}{\hbar}S[x,g]}$$

over maps $x : \Sigma \to E$ and metrics g on Σ. It will be *Euclidean*, in all respects: both metrics G on E and g on Σ will be Riemannian and the integral will be Gaussian rather then oscillatory.

1.3. **Wick rotation.** The Wick rotation from the Minkowski space M to the Euclidean space E will be a change of coordinates

$$x^0, ..., x^{D-1} \text{ on } M \ \mapsto \ x^1, ..., x^D \text{ on } E, \quad \text{given by } x^D = ix^0,$$

This will change the metric from $G_M = -(x^0)^2 + \sum_1^d (x^i)^2$ on M, to $G_E = \sum_1^D (x^i)^2$ on E.

One can think of M and E as two real forms of $M_{\mathbb{C}} = \mathbb{C}^n = E_C$. Here $\mathbb{C}^n$ comes with the quadratic form $G_{\mathbb{C}} = \sum z_i^2$, and two real subspaces have the induced metrics: $E = \mathbb{R}^n$ has the Euclidean metric $g_E = \sum_1^D (x^i)^2$, and the Minkowski space $M = i\mathbb{R} \oplus \mathbb{R}^{n-1}$ has the Lorentzian metric $G_M = \eta = G_M = -(x^0)^2 + \sum_1^d (x^i)^2$.

1.3.1. *Path integrals.* The physics is interested in the maps to M, but the mathematical treatment will be to calculate with the maps to E and then use the analytic continuation of formulas. The above Euclidean path integral is precisely the Wick rotation (or analytic continuation) of the Minkowski path integral.

1.4. **Gauge fixing: "the conformal gauge".** We will interpret the path integral as an equivariant integral for the action of the *gauge group* (reparameterizations and the Weyl rescaling)

$$\boldsymbol{\mathcal{G}} \stackrel{\text{def}}{=} Diff_0(\Sigma) \times C^\infty(\Sigma, \mathbb{R}_{>0}),$$

on the fields (x, g).

1.4.1. *Integration using a gauge.* The gauge group $\boldsymbol{\mathcal{G}}$ acts on the space of fields $\mathcal{F}$, and our integral is really on the quotient space $\mathcal{F}/\boldsymbol{\mathcal{G}}$. A *gauge* means a *section*. Here it means that we choose a subspace $\mathcal{F}_{gauge} \subseteq \mathcal{F}$ which plays the role of a section of $\mathcal{F} \to \mathcal{F}/\boldsymbol{\mathcal{G}}$. This means that $\mathcal{F}_{gauge}$ surjects onto the quotient $\mathcal{F}/\boldsymbol{\mathcal{G}}$, hence

$$\mathcal{F}/\boldsymbol{\mathcal{G}} \stackrel{\simeq}{\leftarrow} \mathcal{F}_{gauge}/\boldsymbol{\mathcal{G}}_{gauge}$$

for the groupoid $\mathcal{G}_{gauge}$ of residual symmetries of $\mathcal{F}_{gauge}$. We call this groupoid the *residual gauge freedom*. Of course we want to have a situation where it comes from the action of the stabilizer subgroup $\boldsymbol{\mathcal{G}}_{\mathcal{F}_{gauge}} \subseteq \boldsymbol{\mathcal{G}}$ of the chosen gauge $\mathcal{F}_{gauge}$.

1.4.2. *Conformal gauge.* (See also 3.7.) In our cause the space of fields is a product $\mathcal{F} = Map(\Sigma, M) \times Metrics(\Sigma)$ and by the *conformal gauge* we mean the the subspace of fields $\mathcal{F}_{gauge} = Map(\Sigma, M) \times \{g_{st}\}$, where we fix a complex structure and a holomorphic coordinate z on Σ, and g_{st} is the standard or *fiducial* metric $g_{st} = \widehat{g}$ with $\widehat{g}_{ij} = \delta_{ij}$ for the corresponding real coordinates $\tau^1 = Re(z)$, $\tau^2 = Im(z)$.

This is the same as choosing global real coordinates τ^i, so we are at the moment really assuming that Σ is a disc! So we are in some sense looking at the "Σ-local properties of the path integral".

1.5. **Residual gauge freedom.** We will see that the choice of the conformal gauge reduces the symmetry from the gauge group $\boldsymbol{\mathcal{G}}$ to the group of conformal automorphisms $Conf(\Sigma, \widehat{g}, \sigma)$.

1.5.1. *Conformal maps.* A diffeomorphism $\phi \in Diff(\Sigma)$ is said to be conformal for a metric g if $\phi^* g$ is conformal to g. These form a subgroup $Conf(\Sigma, \widehat{g}, \sigma) \subseteq Diff(\Sigma) \subseteq \boldsymbol{\mathcal{G}}$ of the gauge group.

1.5.2. *Lemma.* (a) Holomorphic automorphisms ϕ of Σ are conformal.

(b) The projection $\boldsymbol{\mathcal{G}} \twoheadrightarrow Diff(\Sigma)$ identifies the residual gauge freedom $\boldsymbol{\mathcal{G}}_{\mathcal{F}_{gauge}}$ with the group $Conf(\Sigma, \widehat{g}, \sigma)$ of conformal maps.

Proof. (a) For a local holomorphic coordinate z, $\phi(z)$ is another local holomorphic coordinate and $(\phi^* g)_{z\bar{z}} = g(d\phi d, d\phi\bar{\partial}) = g(\phi' d, \overline{\phi'}\bar{\partial}) = |\phi'|^2 \cdot g_{z\bar{z}}$. Similarly, $(\phi^* g)_{zz} = (\phi')^2 \cdot g_{zz} = 0 = (\phi^* g)_{\bar{z}\bar{z}}$, hence $\phi^* g = |\phi'|^2 \cdot g$.

(b) Once we have fixed the metric to be $\widehat{g}$, there is still a residual symmetry given by the conformal automorphisms ϕ of $(\Sigma, \widehat{g}, \sigma)$. For such reparameterization ϕ, $\phi^* \widehat{g} = \omega_\phi \cdot g$ for certain function $\omega_\phi > 0$ (the conformal factor of ϕ. So $(\phi, {\omega_\phi}^{-1}) \in \mathcal{G}$ fixes the metric g.

TO THE END OF THIS SECTION IT IS HAZY GENERAL STRATEGY

SEEMINGLY RELATED TO THE NEXT SECTION, SO

GO TO THE NEXT SECTION FIRST

1.6. **From path integrals to operators.** Abstractly, if we have an action functional $S[y]$ on the space of fields Y, we form the (Euclidean) path integral

$$\int_Y dy \; e^{-S[y]},$$

also called the *partition function.*

1.6.1. *Path integral quantization of functions on the space of fields.* The path integral quantization of a function $\mathcal{F}$ on Y consists of the associated *correlations functions.*

First, we define the *expectation* of $\mathcal{F}$ (the expected value) to be

$$\langle \mathcal{F} \rangle \stackrel{\text{def}}{=} \int_Y dy \; e^{-S[y]}.$$

1.6.2. *Insertion at points of* §. Consider the case when Y is the space of some kind of fields on Σ, that have values in a vector space E. Let x^μ) be the coordinates on E.

At each point $a \in \Sigma$ we have *evaluation functionals*

$$ev_a^\mu[x] \stackrel{\text{def}}{=} x^\mu(a).$$

So we can consider the functionals

$$\mathcal{F} \cdot ev_{a_1}^{\mu_1} \cdot \cdots \cdot ev_{a_\psi}^{\mu_\psi}$$

which are usually denoted

$$\mathcal{F}[x] \cdot x^{\mu_1}(a_1) \cdot \cdots \cdot x^{\mu_\psi}(a_\psi).$$

We say that the *correlation function*

$$\langle \mathcal{F}[x] \cdot x^{\mu_1}(a_1) \cdot \cdots \cdot x^{\mu_\psi}(a_\psi) \rangle$$

is obtained by insertions at a_i's.

1.6.3. In the operator quantization of the function $\mathcal{F}$ on Y, one associates to $\mathcal{F}$ an operator $\widehat{\mathcal{F}}$ on a Hilbert space $\mathcal{H}$. Then the above correlation functions are interpreted as matrix coefficients of $\widehat{\mathcal{F}}$ and its compositions with other operators.

1.6.4. *The case of strings.* The evaluation functional

$$ev_a^\mu[x] = \; x^\mu(a)$$

should quantize to operators

$$\widehat{x}^\mu(a).$$

When viewed as functions of $a \in \Sigma$ they form a function from Σ to operators on $\mathcal{H}$,

$$\widehat{x}^\mu(z, \overline{z}).$$

1.6.5. *Weak vanishing of the quantum stress tensor.* The quantization of the stress tensor $\boldsymbol{T}$, i.e., the non-dynamical variation $\frac{\delta S}{\delta \gamma^{ij}}$ is zero.

However, we will find that in the physical Hilbert state this is true in the *weak sense only*!

1.7. **Faddeev-Popov determinant.** We consider the slice of *all* $(x,g)\to$ *all* $(x,g)/\mathcal{G}$ given by the fiducial metric, so the slice consists of all $(\widehat{g},x)$. Then the path integral is $\int\ Dx\ \Delta_{FP}(\widehat{g})\ e^{-S[x,\widehat{g}]}$ Where, Δ_{FP} is the correct (Faddeev-Popov) measure on the slice, i.e.,

$$\Delta_{FP}{}^{-1}\stackrel{\text{def}}{=}\int_{\mathcal{G}} D\zeta\ \delta_{g,{}^{\zeta}\widehat{g}}$$

is the (measure) of gauge transforms ζ such that ${}^{\zeta}\widehat{g}$ hits g, i.e., the difference between the measure in fields (g,x) and the measure in the gauge group

The calculation of the measure involves the Jacobians for the change of variables. Their systematic treatment involves Grassmann variable b,c, i.e., super-mathematics (determinants= $\wedge^{top}\mathbb{R}^n$= super-mathematics).

The resulting calculation of the path integral is is

$$(\det\widehat{\nabla}^2)^{-D/2}\cdot\det(\widehat{P}_1)$$

here the first determinant is $\int\ Dx$ and the second is a ghost integral $\int\ DbDc$.

1.7.1. *Conclusion.* The integral with the fixed gauge is meaningful and can be calculated. The original "invariant" path integral can be treated as supplying the intuition for the second one.

However, in the gauged fixed form, "the essential content of the gauge symmetry" is still present as the *BRST-invariance*.

2. The naive Hilbert space: quantization of criticality with respect to the path

We will quantize the free strings in a Minkowski space (without ever mentioning the Euclidean version).

2.1. **Operator quantization of classical solutions.** We want to quantize the classical solutions into linear operators on a Hilbert space. We consider Σ which is a tube or an annulus, with the coordinates $\tau^1 = t$ (timelike direction), and $\tau^2 = \theta$ (circle direction). So we can think of a map $x : \Sigma \to E$ in terms of its Fourier series $x(t,\theta) = \sum_{\mathbb{Z}} x_n(t)e^{in\theta}$.

We are considering the criticality equations in the space of all pairs (x, g). The criticality in g requires that g is conformal to the pull-back of G, it was encoded into the vanishing of the stress tensor $\boldsymbol{T}$. The criticality in x (requires that x is (G, g)-harmonic, in the conformal gauge (i.e., for the standard metric $\widehat{g}$), this is the standard harmonicity equation $\partial\overline{\partial}\, x_\mu = 0$.

Let us make a difference between the dynamical variable x and a non-dynamical g. While x will give the Hilbert space $\mathcal{H}_{naive}$, g will give a constraint on this space, i.e., it will carve out a piece $\mathcal{H}_{Ph}$ with a physical meaning.

Now we will concentrate on x and its harmonicity equation. It means that x is a sum of a holomorphic and an anti-holomorphic function. In other words ∂x is holomorphic and $\overline{\partial} x$ is anti-holomorphic. We denote the Laurent series expansions in the annulus by

$$\partial x(z,\bar{z}) \stackrel{\text{def}}{=} \sum_{\mathbb{Z}} \alpha_n z^{n-1} \qquad \text{and} \qquad \overline{\partial} x(z,\bar{z}) \stackrel{\text{def}}{=} \sum_{\mathbb{Z}} \widetilde{\alpha}_n \overline{z}^{n-1}.$$

Here, $\alpha_n, \widetilde{\alpha}_n$ are functionals on the space of classical solutions x, but we can actually extend them to functionals on all maps x by integrating over a chosen circle $0\times S^1 \subseteq \mathbb{R}\times S^1 = \Sigma$, i.e., the unit circle $z = e^{i\theta}$ in $\Sigma = \mathbb{C}^* \stackrel{\text{def}}{=} \mathbb{C} - \{0\}$,

$$\alpha_n(x) \stackrel{\text{def}}{=} \int_{S^1} (\partial_z x)(t,\theta)\big|_{t=0}\ z^{-n}\frac{dz}{2\pi i} = \int_0^{2\pi} (\partial_z x)x(0,\theta)\ e^{-i(n-1)\theta}\frac{d\theta}{2\pi},$$

$$\widetilde{\alpha}_n(x) \stackrel{\text{def}}{=} \int_{S^1} (\bar{\partial}_z x)(t,\theta)\big|_{t=0}\ \bar{z}^{-n}\frac{d\bar{z}}{-2\pi i} = \int_0^{2\pi} (\bar{\partial}_z x)x(0,\theta)\ e^{i(n-1)\theta}\frac{d\theta}{2\pi}.$$

These functionals will quantize to operators $\widehat{\alpha}_n, \widehat{\widetilde{\alpha}}_n$ on the Hilbert space $\mathcal{H}_{naive}$.

We will construct $\mathcal{H}_{naive}$ by a standard procedure in physics. We interpret the Hamiltonian of Polyakov action in the conformal gauge as a sum of harmonic oscillators and a free particle. This interpretation gives a Poisson structure and therefore a prescription for the quantization. So the essence of this approach is that the action suggests the Poisson structure.

One could make the same choice of a Poisson structure on standard mathematical grounds. Algebraically, this is the same choice as in the construction of affine Heisenberg Lie algebras $\widehat{\mathfrak{h}}$, i.e., the cocycle giving the central extension $\widehat{\mathfrak{h}}$ of the loop space $\mathfrak{h}[t,t^{-1}]$ by $\mathbb{C}K$. Geometrically, it means that one considers the space of *formal loops* $\mathfrak{h}[t,t^{-1}]$ in $\mathfrak{h}$ as a cotangent bundle to the space of *formal pointed discs* $t\mathfrak{h}[t]$ in $(\mathfrak{h},0)$.

2.2. **The action for the standard metric on a torus.** On a torus $\Sigma = \mathbb{R}\times S^1$, with coordinates t (time) and θ (angle), and the standard (Lorentzian) metric $g_{st} = -\partial_t^2+\partial_\theta^2$, the action is

$$S[x,g_{st}] \stackrel{\text{def}}{=} T\cdot\int_{x(\Sigma)} dA\ \frac{-||dx||^2_{G,g}}{2} = \frac{-T}{2}\int_\Sigma dtd\theta\ (dx)^2 = \frac{T}{2}\int_\Sigma dtd\theta\ x_t^2-x_\theta^2.$$

2.2.1. *When string is s particle.* This is the case when $x(t,\theta)=x(t)$ does not depend on the angle θ. Then the action is

$$S[x,g_{st}] = \frac{T}{2}\int_\Sigma dtd\theta\ x_t^2-x_\theta^2 = \frac{T}{2}\int_{\mathbb{R}}dt\int_0^{2\pi}d\theta\ \ x_t^2 = \frac{2\pi T}{2}\ \int_{\mathbb{R}}dt\int_0^{2\pi}\ x_t^2.$$

When $T=1/2\pi$ this is the Newtonian kinetic energy action for a free particle (of unit mass!).

2.2.2. *Fourier expansion.* We will use the the Fourier expansion in the direction of the string

$$x(t,\theta) = \sum_{\mathbb{Z}} x_n(t)\cdot e^{in\theta},$$

here $x(t,\theta)\in E$ but $x_n(t)\in E_{\mathbb{C}}$ and $\overline{x_n(t)} = x_{-n}(t)$. Now,

$$S[x,g_{st}] = \frac{T}{2}\int_{\mathbb{R}}dt\int_0^{2\pi}d\theta\ \sum_{p,q}\dot{x}_p(t)e^{ip\theta}\underset{G}{\cdot}\dot{x}_q(t)e^{iq\theta} - \sum ip\cdot x_p(t)\cdot e^{ip\theta}\underset{G}{\cdot}iq\cdot x_q(t)\cdot e^{iq\theta}$$

$$= \frac{T}{2}\cdot 2\pi\int_{\mathbb{R}}dt\ \sum_n \dot{x}_n(t)\underset{G}{\cdot}\dot{x}_{-n}(t) + n(-n)\cdot x_n(t)\underset{G}{\cdot}x_{-n}(t) = \int_{\mathbb{R}}dt\ L(x,\dot{x}).$$

The "time Lagrangian" $(x,\dot{x})$ is therefore a sum

$$L(x,\dot{x}) = \sum_n L_n(x_n,\dot{x}_n)\quad L_n(x_n,\dot{x}_n) = T\pi\sum \dot{x}_n\dot{x}_{-n} - n^2x_nx_{-n} = 2T\pi\,(\sum_{n>0}\dot{x}_n\dot{x}_{-n} - n^2x_nx_{-n} + \frac{1}{2}\dot{x}_0^2).$$

2.2.3. *The choice $T = 1/2\pi$ of the string tension.* This choice really means that we use the standard normalized measure on the circle. Now, the Polyakov string action reduces to the Newtonian action for a free particle when the string degenerates to a point, and also

$$L_n(x_n,\dot{x}_n) = \sum_{n>0}\dot{x}_n\dot{x}_{-n} - n^2x_nx_{-n} + \frac{1}{2}\dot{x}_0^2.$$

2.3. **The canonical formalism.** The *canonical momentum* "conjugate to" the coordinate function x_n is

$$p_n \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{x}_n} = \dot{x}_{-n} \underset{G}{\cdot} - .$$

The *Hamiltonian* function H associated to the Lagrangian function $L(x, \dot{x})$ is the kinetic energy minus the Lagrangian:

$$H \stackrel{\text{def}}{=} \langle p_n, \dot{x}_n \rangle - L = \sum_{n>0} 2p_n \underset{G^*}{\cdot} p_{-n} + p_0^2 - (p_{-n} \underset{G^*}{\cdot} p_n - n^2 x_n \underset{G}{\cdot} x_{-n} + \frac{1}{2}p_0^2)$$

$$= \sum_{n>0} p_n \underset{G^*}{\cdot} p_{-n} + n^2 x_n \underset{G}{\cdot} x_{-n} + \frac{1}{2}p_0^2.$$

2.3.1. *Interpretation.* We will understand the contributions in the following way:

- $\frac{1}{2}p_0^2$ is the Hamiltonian of a free particle (of unit mass), it corresponds to the motion of the center of the mass of the string.
- $p_n p_{-n} + n^2 x_n x_{-n}$ is a Hamiltonian of a pair of harmonic oscillators. It corresponds to the n^{th} mode of oscillations of the string around the fixed center of mass.

HERE (After this, go back and finish II.3.5.9 .)

2.4. **REVISION ENDS HERE**.

2.4.1. *Recollections on harmonic oscillators.* In a vector space M with a metric G, we consider a harmonic oscillator with the frequency ω. Its Hamiltonian is

$$H = \sum_i^n p_i^2 + \omega^2 q_i^2$$

We will think of it as a new inner product

$$(p', q') \cdot (p'', q'') = p_i' p_i'' + \omega^2 q_i' q_i''$$

on $T^*M = T^*\mathbb{R}^n$. The corresponding Poisson structure on T^*M is

$$\{p_i, q_j\} = \delta_{ij}.$$

Now, in order to think of the quadratic form $H(p_+, p_-, q_+, q_-) = p_+ p_- + \omega^2 q_+ q_-$ as a pair of oscillators we change the variables from $a_\varepsilon = (p_\varepsilon, q_\varepsilon)$, $\varepsilon = \pm$, to $a^\varepsilon = \frac{1}{2}(a_+ + \varepsilon a_-)$, so that i.e., $a_\varepsilon = (a^+ + \varepsilon a^-)$ and

$$p^\varepsilon = \frac{1}{2}(p_+ + \varepsilon p_-), \quad q^\varepsilon = \frac{1}{2}(q_+ + \varepsilon q_-), \quad i.e., \quad p_\varepsilon = p^+ + \varepsilon p^-, \quad q_\varepsilon = q^+ + \varepsilon q^-.$$

Then we can interpret $H = a_+ \cdot a_- = (\frac{a_+ + a_-}{2})^2 - (\frac{a_+ - a_-}{2})^2 = (a^+)^2 + (ia^-)^2$ as a pair of harmonic oscillators and with this interpretation the Poisson structure is $\{p^\varepsilon, q^\varepsilon\} = \varepsilon$ and $\{p^\varepsilon, q^{-\varepsilon}\} = 0$. Therefore, $\{p_\varepsilon, q_{\varepsilon'}\} = \{p^+ + \varepsilon p^-, q^+ + \varepsilon' q^-\} = 1 - \varepsilon\varepsilon'$, hence,

$$\{p_\varepsilon, q_{-\varepsilon}\} = 2, \qquad \text{and} \qquad 0 = \{p_\varepsilon, q_\varepsilon\} = \{p_+, p_-\} = \{q_+, q_-\}.$$

We will apply this to $H_n = p_n p_{-n} + \omega^2 x_n x_{-n}$ with $p_+ = p_n$, $q_+ = x_{-n}$ and $p_- = p_{-n}$, $q_- = x_n$!

[The correct framework should be the notion of a harmonic oscillator in a vector space M (with a metric G), and with the frequency ω. It is given by the Hamiltonian $H = p^2 + \omega^2 q^2 \stackrel{\text{def}}{=} p \underset{G}{\cdot} p + \omega^2 q \underset{G}{\cdot} q$. We can think of it as an inner product $(p', q') \cdot (p'', q'') = p' \underset{G}{\cdot} p'' + \omega^2 q' \underset{G}{\cdot} q''$ on $T^*\mathbb{M}^n$.]

2.4.2. *The change of variables.* So, our Hamiltonian suggests the Poisson structure in which p_n's commute and x_n's commute, and $\{(p_\mu)_n, (x^\nu)_m\} = 2\delta_{n,m}\delta_{\mu\nu}$, which we denote symbolically by $\{p_n, x_m\} = 2\delta_{n,m}$. The same as for the harmonic oscillator, we pass to complex coordinates. In terms of quantities

$$\alpha_n = \frac{1}{2}(p_n - inx_{-n}), \qquad \text{left movers,}$$

$$\widetilde{\alpha}_n = \frac{1}{2}(p_{-n} - inx_n), \qquad \text{right movers,}$$

the Poisson structure is

$$\{\alpha_n, \alpha_m\} = in\delta_{n+m} = \{\widetilde{\alpha}_n, \widetilde{\alpha}_m\} \qquad \text{and} \qquad \{\widetilde{\alpha}_n, \alpha_m\} = 0.$$

(One has, $\{\alpha_n, \alpha_m\} = (\frac{1}{2})^2 \cdot 2 \cdot \{p_n, -imx_{-m}\} = \frac{1}{2}\delta_{n,-m} \cdot (-im) \cdot 2$, etc.) We will also record the inverse change of variables: since $\alpha_{-n} = \frac{1}{2}(p_{-n} + inx_n)$, $\widetilde{\alpha}_{-n} = \frac{1}{2}(p_n + inx_{-n})$, one has

$$p_n = \alpha_n + \widetilde{\alpha}_{-n}, \quad x_n = \frac{1}{in}(\alpha_{-n} - \widetilde{\alpha}_n) = \frac{i}{n}(\widetilde{\alpha}_n - \alpha_{-n}).$$

One has $\overline{\alpha_n} = \alpha_{-n}$ and $\overline{\widetilde{\alpha}_n} = \widetilde{\alpha}_{-n}$, since $x_{-n} = \overline{x_n}$ and the same for $p_n = \dot{x}_{-n}$.

2.5. **Quantization of functions.** The functions $\alpha_n, \widetilde{\alpha}_n$ quantize to operators $\widehat{\alpha}_n, \widehat{\widetilde{\alpha}}_n$ (we often forget the hat). The prescription for the quantization of Poisson brackets is that $\{f, g\} = h$ gives $\{\widehat{f}, \widehat{g}\} = \frac{\hbar}{i}\widehat{h}$, hence

$$[\alpha_n, \alpha_m] = n\hbar\delta_{n+m} = [\widetilde{\alpha}_n, \widetilde{\alpha}_m] \qquad \text{and} \qquad [\widetilde{\alpha}_n, \alpha_m] = 0.$$

Also,

$$[\widehat{p}_n, \widehat{x}_m] = 2\frac{\hbar}{i}\delta_{n,m} \qquad \text{and} \qquad 0 = [p_n, p_m] = [x_n, x_m].$$

We are usually interested in the Planck scale, i.e., $\hbar = 1$.

2.6. **The Hilbert space.** We will see that we have no choice in the choice of the Hilbert space for the oscillations – the commutators of $\alpha_n, \widetilde{\alpha}_n,\ n > 0$, forces it to be the oscillator representation of the affine Heisenberg Lie algebra. The remainder is the standard quantization of the free particle part (i,e., $\alpha_0 = \widetilde{\alpha}_0 = \frac{1}{2}p_0$).

(1) Adjointness conditions. Recall that $\overline{\alpha_n} = \alpha_{-n}$ and $\overline{\widetilde{\alpha}_n} = \widetilde{\alpha}_{-n}$. This quantizes to $\alpha_n^* = \alpha_{-n}$ and $\widetilde{\alpha}_n^* = \widetilde{\alpha}_{-n}$, where $*$ denotes the Hermitian adjoint.

(2) Operators generate affine Heisenberg Lie algebra. The above commutators show that α's and $\widetilde{\alpha}$'s span two commuting copies of the the affine Heisenberg Lie algebra $\widehat{M} = M[t,t^{-1}] \oplus \mathbb{C}K$ corresponding to the metric G on M. More precisely, the condition $\hbar = 1$ means that α_n's generate the action of the enveloping algebra $U(\widehat{M})$ in which K acts by 1.

(3) Unitary representations of affine Heisenberg Lie algebras at $K = 1$. Recall that one can "split off the zero mode" from the affine Heisenberg Lie algebra, i.e., it decomposes into a sum of Lie algebras $\widehat{M} = M' \oplus M$. Since M is abelian, $U_{K=1}(\widehat{M}) = U_{K=1}(M') \otimes S[M]$, and we recall that $U_{K=1}(M')$ is the algebra $D_{\mathcal{V}}$ of differential operators on a certain vector space $\mathcal{V}$.

The only irreducible representations of the affine Heisenberg Lie algebra $\widehat{M}$ in which K acts by 1 is the so called oscillator representation Osc_M. We have constructed it as a Verma module and identified it with the space $\mathcal{O}(\mathcal{V})$ of polynomial functions on the vector space $\mathcal{V} = \oplus_{k<0}\, t^k M$.

Moreover, if a representation of $U_{K=1}(M')$ on a Hilbert space satisfies the adjointness conditions above (we call such representations unitary), it is a multiple of the oscillator representation.

(4) Unitary representations of α_n's and $\widetilde{\alpha}_n$'s. To put everything together, we introduce another copy $\widetilde{M}$ of M and observe that α's and $\widetilde{\alpha}$'s generate an action of $U_{K=1}(\widehat{(M \oplus \widetilde{M})}) = U_{K=1}(M \oplus \widetilde{M})' \otimes\ S(M \oplus \widetilde{M})$. However, the 0-modes in the two copies $\widehat{M}$ coincide since we had only one operator $\alpha_0 = \widetilde{\alpha}_0$ with the index 0. So the action of $S(M \oplus \widetilde{M})$ reduces to $S(M)$.

Now the part (3) (applied to $M \oplus \widetilde{M}$), shows that our Hilbert space is the tensor product of the oscillator representation $Osc_{M \oplus \widetilde{M}}$ of $(M \oplus \widetilde{M})'$ and a representation $\mathcal{H}_0$ of the abelian Lie algebra M! (When viewed as a representation of $U_{K=1}(M \oplus \widetilde{M})'$ only, then $\mathcal{H}_0$ is the multiplicity of the oscillator representation $Osc_{M \oplus \widetilde{M}}$.)

Moreover, $Osc_{M \oplus \widetilde{M}} = Osc_M \otimes Osc_{\widetilde{M}}$ (clearly $\oplus_{k<0}\, t^k(M \oplus \widetilde{M}) = \oplus_{k<0}\, t^k M \oplus \oplus_{k<0}\, t^k(\widetilde{M})$ gives $\mathcal{O}(\oplus_{k<0}\, t^k M) \otimes \mathcal{O}(\oplus_{k<0}\, t^k(\widetilde{M})) \stackrel{\cong}{\leftarrow} \mathcal{O}(\oplus_{k<0}\, t^k(M \oplus \widetilde{M}))$.)

(5) The Hilbert space. Finally, the representation $\mathcal{H}_0$ is the quantization of $H_0 = \frac{1}{2}p_0^2$ of a free particle, so it is $L^2(M)$. So

$$\mathcal{H} = Osc_M \otimes Osc_{\widetilde{M}} \otimes L^2(M).$$

Here, The action of M on L^2M is the quantization of velocity (momentum) of a particle. L^2M has an idealized basis of pure momentum states $|\lambda_m\rangle$, $\lambda \in M^*$ (as a function $|\lambda_m\rangle = e^{i\langle\lambda|-\rangle}$). On the corresponding line $\mathbb{C}|\lambda_m\rangle$ M acts by λ, and l^2M is a continuous sum – an integral – of these one-dimensional representations $\int_{M^*} d\lambda\ \mathbb{C}|\lambda\rangle$.

(Physicists sometimes talk of $L^2(M)$ as a sum of these lines, hence of $\mathcal{H}$ as a sum of Hilbert spaces $Osc_M \otimes Osc_{\widetilde{M}} \otimes \mathbb{C}|\lambda\rangle$.)

2.7. **The Hamiltonian.** We will quantize the Hamiltonian using the symmetric quantization in p, x-coordinates. Then we will recalculate it in terms of $\widehat{\alpha}_n, \widehat{\widetilde{\alpha}}_n$ in such way that positive n's are on the right (normal ordering).

For each n,

$$(p_n p_{-n} + n^2 x_n x_{-n})\widehat{\ } \stackrel{\text{def}}{=} \frac{1}{2}(\widehat{p}_n\widehat{p}_{-n} + n^2\widehat{x}_{-n}\widehat{x}_n) + \frac{1}{2}(\widehat{p}_{-n}\widehat{p}_n + n^2\widehat{x}_n\widehat{x}_{-n})$$

$$= \frac{1}{2}[(\widehat{p}_n + in\widehat{x}_{-n})(\widehat{p}_{-n} - in\widehat{x}_n) - (in\widehat{x}_{-n}\widehat{p}_{-n} - in\widehat{p}_n\widehat{x}_n)] + \frac{1}{2}[(\widehat{p}_{-n} + in\widehat{x}_n)(\widehat{p}_n - in\widehat{x}_{-n}) - (in\widehat{x}_n\widehat{p}_n - in\widehat{p}_{-n}\widehat{x}_{-n})]$$

$$= \frac{1}{2}[\widehat{\widetilde{\alpha}}_{-n}\widehat{\widetilde{\alpha}}_n + \widehat{\alpha}_{-n}\widehat{\alpha}_n + in[\widehat{p}_n, \widehat{x}_n] + in[\widehat{p}_{-n}, \widehat{x}_{-n}]].$$

The sum over $n > 0$ gives a meaningless formula

$$\widehat{H} = \frac{1}{2}p_0^2 + \sum_{n>0} \frac{1}{2}\widehat{\widetilde{\alpha}}_{-n}\widehat{\widetilde{\alpha}}_n + \frac{1}{2}\widehat{\alpha}_{-n}\widehat{\alpha}_n + 2n.$$

2.7.1. *ζ-regularization.* Recall that $\zeta(s) \stackrel{\text{def}}{=} \sum_{n>0} n^{-s}$ converges for $Re(s) > \frac{1}{2}$ and has an analytic continuation to the complex plane. So we can interpret $\sum_{n>0} n$ as $\zeta(-1) = -1/12$.

Then $\widehat{H} \stackrel{\text{def}}{=} H_L + H_R$ for

$$H_L \stackrel{\text{def}}{=} \frac{1}{4}p_0^2 + \frac{1}{2}\sum_{n>0} \widehat{\alpha}_{-n}\widehat{\alpha}_n + \zeta(-1), \quad H_R \stackrel{\text{def}}{=} \frac{1}{4}p_0^2 + \frac{1}{2}\sum_{n>0} \widehat{\widetilde{\alpha}}_{-n}\widehat{\widetilde{\alpha}}_n + \zeta(-1).$$

3. The String Spectrum

Recall that we still have to impose the criticality in g, i.e., the vanishing if the stress tensor.

Moreover, there is one more problem from the point of view of physics – in the so called naive Hilbert space $\mathcal{H}_{naive}$ that we have constructed, there is a canonical non-degenerate hermitian form compatible with the action of the operators $\widehat{\alpha}_n, \widehat{\widetilde{\alpha}}_n$ (in the sense that $\widehat{\alpha}_n^* = \widehat{\alpha}_{-n}$). However, it is not positive definite so that $\mathcal{H}_{naive}$ is not exactly a Hilbert space. This problem arises because the construction of the affine Heisenberg algebra uses metric G on M and for physics requires that G be indefinite (Euclidean G gives directly a Hilbert space). This contradicts the physical intuition – the norm should be a probability, hence ≥ 0.

It turns out that the solution of the first problem also solves the second. The first problem forces the dimension of the spacetime to be 26 (called the "critical dimension" for the bosonic string or for the representation theory of the Virasoro Lie algebra).

3.0.2. Recall that in conformally flat coordinates the stress tensor is given by a holomorphic function $\boldsymbol{T}(z)(x,g) \stackrel{\text{def}}{=} \boldsymbol{T}_{zz}(x,g) = \partial_z x \underset{G}{\cdot} \partial_z x$ and its conjugate $\widetilde{\boldsymbol{T}}(\bar{z})(x,g) \stackrel{\text{def}}{=} \boldsymbol{T}_{\bar{z}\bar{z}}(x,g)$. We develop these into Laurent series

$$\boldsymbol{T}(z) = \sum L_m z^{-n-2}, \quad \widetilde{\boldsymbol{T}}(\bar{z}) = \sum L_m \bar{z}^{-n-2}.$$

4. Appendix A. The Weyl anomaly

This is one more train of thought leading to the conclusion $D = 26$. One considers the conformal invariance on the quantum level in the language of physics (stress tensor).

The Weyl anomaly is the fact that in order to have a consistent theory one needs some assumption on the background (M, G). For a bosonic string in the flat (Minkowskian) spacetime, the condition is $D = 26$.

The problem appears as $\boldsymbol{T}^i{}_i \stackrel{\text{def}}{=} \gamma^{ij}\boldsymbol{T}_{ji}$ vanishes in the classical theory (Weyl invariance), but not in the q-theory unless $D = 26$. So the q-theory is Weyl invariant only for $D = 26$.

Part VI. Representations as the algebraic structure on the Hilbert space of a quantum string

0. Intro

0.1. **Hilbert space of the bosonic string.** Our goal is to describe in a mathematically natural way the Hilbert space $\mathcal{H}_{string}(M,G)$ of a bosonic string in a Riemannian manifold (M,G). The basic case we consider is when (M,G) is the Minkowski space (we will also sketch the case when M is a compact Lie group U with an invariant metric, and we will hint towards the general case). Since physics is interested in the Minkowski metric, our G need not be positive definite.

In consequence, the naive construction produces a larger vector space $\mathcal{H}_{naive}$ with a natural hermitian inner product which is not positive definite. Then one cuts out the bad part and obtains the above true Hilbert space $\mathcal{H}_{string}(M,G)$. For now we will only be interested in constructing the naive version which we will call $\mathcal{H}$.

0.2. **The structure of $\mathcal{H}$ (phantasy).** In, general, if we are quantizing "evolutions of the string of a topological type Σ", i.e., maps from a surface Σ to (M,G), then $\mathcal{H}$ will depend on M and the choice of S^1-parameterizations ι_S of loops S in $\partial\Sigma$.

The fundamental object is a bundle of Hilbert spaces $\widetilde{\mathcal{H}}_M(\Sigma,\iota) \to M$. Its fibers $\mathcal{H}_{M,m}(\partial\Sigma,\iota)$ are multiplicative in the set $\pi_0(\partial\Sigma)$ of connected components of the boundary $\partial\Sigma$. Precisely, the fiber is a tensor product

$$\mathcal{H}_{M,m}(\partial\Sigma,\iota) = \underset{S\in\pi_0(\partial\Sigma)}{\otimes} \mathcal{H}_{M,m}(S,\iota_S)$$

over loops S in $\partial\Sigma$. In the basic case of open strings, Σ is a strip hence $\partial\Sigma \cong S^1$ and there is one factor. For closed strings Σ is a cylinder, hence $\partial\Sigma \cong S^1 \sqcup S^1$ and there are two factors.

The Hilbert space should then be

$$\mathcal{H}_M(\partial\Sigma,\iota) = \int_M d_G vol(m)\ \ \widetilde{\mathcal{H}}_{M,m}(\partial\Sigma,\iota),$$

However, the physicists talk of a sum $\mathcal{H}_M(\partial\Sigma,\iota) = \oplus_{m\in M}\ \ \widetilde{\mathcal{H}}_{M,m}(\partial\Sigma,\iota)$.

Our goal here is the construction of basic ingredients – vector spaces $\mathcal{H}_{M,m}(S,\iota_S)$, i.e., for each point $m\in M$ we need one vector space

$$\mathcal{H}_m \overset{\text{def}}{=} \mathcal{H}_{M,m}(S^1, id_{S^1}).$$

Moreover, as we will only consider the case when M is a group, we can assume that $m=1$.

0.3. **Representation theory.** Spaces $\mathcal{H}_m$ will carry an action of operators $\widehat{x}_n$ which are quantizations of Fourier modes $x(t,\theta) = \sum x_n(t)e^{in\theta}$ of a moving string. These operators

satisfy algebraic relations $[\widehat{x}_n, \widehat{x}_m] = i\hbar n\delta_{m+n}$, so they generate an algebraic structure acting on $\mathcal{H}$. This structure is usually viewed as a representation of an affine Lie algebra.

In the simplest case of strings in a vector space (say the Minkowski space M), we will also view $\widehat{x}_n$'s as generating the action of differential operators on a space of discs in a Minkowski space. In this case, the mathematical picture of quantizing a string is quite parallel to the quantization of the particle phase space – the cotangent bundle $T^*\mathcal{C}$ to the configuration space $\mathcal{C}$. Here, the string phase space is the space of (based) loops. This is a vector space which one can canonically view as the cotangent bundle of the subspace of (based) "discs" – loops that extend across 0 to a map from a disc. Then the functions on based loops deform to differential operators on based discs.

In the case of strings in a compact semisimple group U, $\mathcal{H}_1$ will carry an action of the corresponding affine Lie algebra $\widehat{\mathfrak{u}}$. Moreover, this action integrates to an action of the corresponding affine group $\widehat{U}$.

0.4. **Loop groups.** The configuration space for strings in a real manifold M is the space of smooth maps $\Lambda(M) \stackrel{\text{def}}{=} C^\infty(S^1, M)$ called the loop space of M. The tangent space at a loop x is the space of sections of the pull-back vector bundle x^*TM over S^1.

The basic case of interest will be when M is a real vector space (the Minkowski space), and more generally, when M is real Lie group U. Then $\Lambda(U)$ is a group for pointwise multiplication of loops, we call such groups loop groups.

Its Lie algebra is the tangent space at a the constant loop 1, i.e., the space $\Lambda(\mathfrak{u})$ of maps into $\mathfrak{u} = T^1U$, with the pointwise Lie bracket $[f, g](t) = [f(t), g(t)]$. It contains a dense Lie subalgebra of polynomial loops $\Lambda_{pol}(\mathfrak{u}) = \{finite\ sums\ \sum_{\mathbb{Z}} a_n e^{in\theta},\ a_n \in \mathfrak{u}\}$.

0.5. **Quantization of based loops.** We are interested in quantizing the space of loops $\Lambda(M)$, i.e., attaching to it a Hilbert spaces $\mathcal{H}_m$. Though we think of the space of loops $\Lambda(M)$ as the configuration space, it is also a "phase space", in the sense that a choice of a metric G on M should make it close to a Poisson manifold – the fibers of the evaluation at 1, $\Lambda(M, m) \stackrel{\text{def}}{=} \{f \in \Lambda(M),\ f(1) = m\}$, $m \in M$, seem to be Poisson spaces. Each of these should contribute a vector space $\mathcal{H}_m$.

We will assume that M is a Lie group U. In this case we will first sketch the construction of $\mathcal{H}_1$ (for $1 \in U$), in a geometric setting, and then we will do a more detailed treatment in an algebraic setting.

If U is a vector space then $\Omega(U)$ is the cotangent bundle to the subspace $\Omega_{\geq 0}(U)$ of "discs", i.e., loops f that extend across 0 in the sense that the Fourier expansion $f = \sum f_n e^{in\theta}$ is a sum over $n \geq 0$ only. Then $\mathcal{H}_1$ is the space of functions on the subspace $\mathcal{D}_0$.

We will also be interested in the case when U is compact. For the generalization to this case we should review the vector space case. For a vector space V it is better to think of functions on $\Omega_{\geq 0}(V)$ as functions on the whole space $\Omega(V)$ with the special property (say,

the property that they are invariant under translations by the complementary subspace $\Omega_{<0}(V)$).

For a compact group U, $\Omega(U)$ will be a complex manifold and $\mathcal{H}_1$ will consist of "smooth functions on $\Omega(U)$ with a special property that they are holomorphic". Actually, this does not quite work since we only get the constant functions. So we modify the formulation: $\mathcal{H}_1 =$ holomorphic sections of a line bundle $\mathcal{L}$ over $\Omega(U)$.

1. Geometry

The (only) source is **Pressley-Segal,** *Loop groups.*

Here we sketch the construction of the Hilbert space $\mathcal{H}_1$ for strings moving in a (semisimple) compact Lie group U.

1.0.1. *On the choice of the geometric setting.* So far, we have been considering the smooth maps from S^1 to a real manifold M. For a moment we let M be a vector space for simplicity. However, there are other meaningful notions of "loops", say, smooth loops in M contain the polynomial loops in M – these are the finite sums $\sum_{n\in\mathbb{Z}} m_n e^{in\theta}$, $m_n \in M$. These can be identified with the algebro-geometric loops $M[t,t^{-1}] =$ polynomial maps from $\mathbb{C}^*$ to M. Moreover, these are now included in the space of "infinitesimal loops " $M((t))$ which are maps from an infinitesimal punctured disc to M.

It turns out that all these versions are often equivalent in the sense that a construction for one kind has analogues for all other kinds. So we will occasionally switch from one of these "languages" to another.

1.1. **Kähler structure on based loops.** When U is a (semisimple) compact Lie group U the group of "based" loops $\Omega(U) = \{f \in \Lambda(U),\ f(1_{S^1}) = 1_U\}$ has an extra structure – it is a Kähler manifold. This means that it is a complex manifold with a real symplectic structure satisfying some properties.

Since $\Omega(U)$ is a group it suffices to describe these structures on the tangent space at 1, i.e., on the Lie algebra $\Omega(\mathfrak{u}) = \{f \in \Lambda(\mathfrak{u}),\ f(1_{S^1}) = 0\}$. In order to $\Omega(\mathfrak{u})$ into a symplectic topological vector space one does not need U to be compact, what is needed is just a symmetric non-degenerate bilinear form $(-,-)$ on $\mathfrak{u}$ which is $\mathfrak{u}$-invariant, i.e., $([x,u],v)+(u,[x,v])=0$ for $u,v,x \in \mathfrak{u}$. The symplectic form is then

$$\omega(f,g) \stackrel{\text{def}}{=} \int_{S^1} (df,g), \quad f,g \in \Omega(\mathfrak{u}).$$

1.2. **Affine groups.** Algebraically, symplectic form ω leads to an extension of Lie algebras

$$0 \to \mathbb{R}\cdot K \to \mathfrak{u}' \to \Omega(\mathfrak{u}) \to 0.$$

Here, $\mathfrak{u}' = \Omega(\mathfrak{u})\oplus\mathbb{R}\cdot K$ as a vector space. While K is central (i,e, $[K,\mathfrak{u}'] = 0$), for $f, g \in \Omega(\mathfrak{u})$ the bracket is

$$[f,g] \stackrel{\text{def}}{=} [f,g]_{\Omega(\mathfrak{u})} + \omega(f,g)\cdot K$$

(the first term is the pointwise bracket in $\Omega(\mathfrak{u})$). On a larger vector space $\Lambda(\mathfrak{u})$ form ω is a skew form but it is degenerate – constant loops $\mathfrak{u}$ are orthogonal to $\Lambda(\mathfrak{u})$. Still, as above, it defines a larger extension of Lie algebras $0\to \mathbb{R}\cdot K\to \widehat{\mathfrak{u}}\to \Lambda(\mathfrak{u})\to 0$ which contains the preceding one.

Back in geometry, if U is connected and simply connected, these extensions of Lie algebras give extensions of Lie groups

$$1\to S^1\to \widehat{U}\to \Lambda(U)\to 1$$

and $1\to S^1\to U'\to \Omega(U)\to 1$. If U is not simply connected, say U is a torus $(S^1)^n$, the central extension of groups require a bit more – a refinement of the form $(-,-)$ to integers.

These extensions of loop groups are called affine groups.

1.3. **Line bundle $\mathcal{L}$ over based loops.** Affine group U' maps to the loop group $\Omega(U)$ and the fibers are circles. More precisely, the map $U'\to \Omega(U)$ is a principal S^1-bundle. If one replaces circle S^1 by the complex line $\mathbb{C}$, one obtains a complex line bundle (vector bundle of rank 1) over based loops

$$\mathcal{L} \stackrel{\text{def}}{=} U' \underset{S^1}{\times} \mathbb{C} \stackrel{\text{def}}{=} S^1\backslash(U'\times \mathbb{C}), \quad for\ s\cdot(u',z) \stackrel{\text{def}}{=} (u'\cdot s^{-1}, s\cdot Z).$$

1.4. **Complex structures on based loops.** If U is a semisimple compact Lie group then $\Omega(U)$ has canonical structures of a complex manifold and the line bundle $\mathcal{L}$ is a holomorphic vector bundle over the complex manifold $\Omega(U)$.

Complex structures on the real vector space $\Omega(\mathfrak{u})$.

1.5. **Approximation of the space of based smooth loops by an algebraic object.** Assume that U is a real algebraic group, i.e., that it is given systems real numbers satisfying some polynomial equations. Then one can complexify U to a complex Lie group $G = U_{\mathbb{C}}$ which consists of complex solutions of the same equations. Say the complexification of the group $U = U(n)$ of unitary matrices is the group $G = GL_n(\mathbb{C})$, case $n = 1$ is $(S^1)_{\mathbb{C}} = \mathbb{C}^*$. The loop Grassmannian of G is $\mathcal{G} \stackrel{\text{def}}{=} G(\mathcal{K})/G(\mathcal{O})$ where $\mathcal{K} = \mathbb{C}[z,z^{-1}]$ and $\mathcal{O} = \mathbb{C}[z]$, however if we use larger rings $\mathcal{K} = C((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$ the homogeneous space remains the same.

Now, let us approximate the space of based loops $\Omega(U)$ by polynomial based loops $\Omega_{pol}(U)$. This is a dense part of $\Omega(U)$ and the inclusion $\Omega_{pol}(U)\subseteq \Omega_{pol}(U)$ is in some sense close to equality – it is a homotopy equivalence. If a loop $x : S^1\to U$ is polynomial loop one can plug in complex numbers, so it extends to a polynomial map from $\mathbb{C}^*$ to G. This gives a map $\Omega_{pol}(U)\to G(\mathbb{C}[t,t^{-1}])$ of groups. It turns out that the composition

$\Omega_{pol}(U) \to G(\mathbb{C}[t,t^{-1}]) \to \mathcal{G}$ is a bijection. So, $\Omega(U)$ can be approximated by a dense subspace $\mathcal{G}$.

1.6. **The Hilbert space $\mathcal{H}_1$.** This is the space of holomorphic sections of the line bundle $\mathcal{L} \to \Omega(U)$. It carries an action of the affine group U', and also of the larger group $\widehat{U}$. Once we differentiate it we get an action of the Lie algebra $\Lambda(\mathfrak{u})$ and its complexification $\Lambda(\mathfrak{g})$. In particular, for $x \in \mathfrak{u}$ and $n \in \mathbb{Z}$, the element $x_n \stackrel{\text{def}}{=} xt^n$ of $\Lambda(\mathfrak{u}) \subseteq \widehat{\mathfrak{u}}$ acts on $\mathcal{H}$. These are the operators one meets in the standard description of the quantization of a bosonic string.

In the algebraic version, one looks at $\mathcal{L}$ as an algebraic line bundle over the loop Grassmannian $\mathcal{G}$ and $\mathcal{H}$ is the space of algebraic (polynomial) sections.

2. Algebra

The idea here is that one constructs representations of some Lie algebra $\mathfrak{g}$ by induction from a Lie subalgebra. The induction construction is based on attaching canonically to each Lie algebra $\mathfrak{a}$ an associative algebra $U\mathfrak{a}$ ("enveloping algebra of $\mathfrak{a}$").

In the particular case when $\mathfrak{g}$ has a "triangular decomposition", induction procedure gives us a nice class of the so called Verma modules V_λ of $\mathfrak{g}$, and each Verma leads to an irreducible representation L_λ - its unique irreducible quotient. This includes the case of Heisenberg Lie algebras associated to symplectic vector spaces and the case of affine Lie algebras.

Recall that our goal is to construct the Hilbert space $\mathcal{H}_1$ for strings in a real Lie group U. For that, $\mathfrak{g}$ will be an affinization $\widehat{\mathfrak{u}_{\mathbb{C}}}$ of the complexification $\mathfrak{u}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{u}$ of U. One of the irreducibles L_λ will be the desired quantization $\mathcal{H}_1$ of the space of based loops $\Omega(U)$.

We will first study affine Lie algebras $\widehat{\mathfrak{u}_{\mathbb{C}}}$ in the case when U is a vector space, these are the so called affine Heisenberg Lie algebras. Then we will mention that this algebraic construction works the same in the case when U is a compact semisimple group. This is the case studied above geometrically, so we obtain an algebraic construction of the space $\mathcal{H}_1$ constructed above as the space of sections of a line bundle.

2.1. **Lie algebras.** The intuition about Lie algebras comes first from their relation to groups. For any Lie group G, the tangent space $\mathfrak{g} \stackrel{\text{def}}{=} T_1 G$ has a structure of a Lie algebra. Lie algebra structure on $\mathfrak{g}$ measures the non-commutativity of G and under some topological conditions it captures the behavior of G. If G is connected and simply-connected then the finite dimensional representations of G and $\mathfrak{g}$ are the same.

2.2. **From Lie algebras to associative algebras.** Let us say that a linearization of Lie algebra $\mathfrak{g}$ is a pair (A, ϕ) consisting of an associative algebra A endowed with a map

of Lie algebras $\phi: \mathfrak{g} \to A$ (i.e., $\phi[x,y] = [\phi x, \phi y]$ for $x, y \in \mathfrak{g}$, where the bracket on the RHS is the commutator in A).

one associates associative algebra $U\mathfrak{g}$ with 1.

2.2.1. *Lemma.* Any Lie algebra $\mathfrak{g}$ has a universal linearization $(U\mathfrak{g}, \iota)$. (Universality means that for any linearization (A, ϕ) there is a unique map of associative algebras $U\mathfrak{g} \xrightarrow{f} A$ such that $\phi = f \circ \iota$.

2.2.2. *Corollary.* (a) Construction $\mathfrak{g} \mapsto U\mathfrak{g}$ is a functor: a map of Lie algebras $\mathfrak{b} \xrightarrow{\alpha} \mathfrak{g}$ defines a map of associative algebras $U\mathfrak{b} \xrightarrow{A} U\mathfrak{g}$.

(b) Representations (=modules) are the same for $\mathfrak{g}$ and $U\mathfrak{g}$, i.e., there are canonical equivalences of categories $Rep(\mathfrak{g}) \cong Rep(U\mathfrak{g})$.

2.2.3. *Lemma.* [Poincare-Birkhoff-Witt] Any ordered basis $x_1, x_2, ...$ of $\mathfrak{g}$ gives a monomial basis $x^e = x_1^{e_1} x_2^{e_2} \cdot \cdots$ of $U\mathfrak{g}$.

2.2.4. *Corollary.* (a) $\mathfrak{g}$ is a subspace of $U\mathfrak{g}$. It generates $\mathbb{U}\mathfrak{g}$ as associative algebra.

(b) For any Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, associative algebra $U\mathfrak{b}$ is a subalgebra of $U\mathfrak{g}$.

(c) Suppose that as a vector space $\mathfrak{g}$ decomposes into a sum $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}$ for Lie subalgebras $\mathfrak{b}, \mathfrak{c}$ of $\mathfrak{g}$. Then the multiplication in $U\mathfrak{g}$ gives an isomorphism of vector spaces $U\mathfrak{b} \otimes U\mathfrak{c} \to U\mathfrak{g}$.

Proof. (b) Choose bases $x_1, x_2, ...$ of $\mathfrak{b}$ and a basis $y_1, y_2, ...$ of a complement to $\mathfrak{b}$ in $\mathfrak{g}$.

2.3. **Heisenberg Lie algebras.**

2.3.1. *The Heisenberg Lie algebra of a symplectic vector space.* Let ω be a skew symmetric bilinear form on a vector space A. On $\widetilde{A} \stackrel{\text{def}}{=} A \oplus \mathbb{C} \cdot K$ there is a canonical structure of a Lie algebra such that K is central, i.e., $[K, \widetilde{A}] = 0$, and $[a, b] = \omega(a, b) \cdot K$ for $a, b \in A$.

Suppose that ω is symplectic, i.e., that it is non-degenerate. An example is $A = T^*V = V \oplus V^*$ with $\omega(v + v^*, u + u^*) = \langle v | u^* \rangle - \langle u | v^* \rangle$.

A polarization of a symplectic space (A, ω) is a decomposition $A = U \oplus V$ into isotropic subspaces (i.e., $\omega = 0$ on U and on V). Then $U \xrightarrow{\cong} V^*$ by $u \mapsto \omega(u, -)$, hence $A \cong T^*V$ (at least if V is finite-dimensional).

2.3.2. *Lemma.* Denote $\mathfrak{g} = \widetilde{A}$. Let us us consider the enveloping algebra $U\mathfrak{g}$ as a family of algebras

$$U_\kappa \mathfrak{g} \stackrel{\text{def}}{=} U(\mathfrak{g})/(K - \kappa \cdot 1) = U(\mathfrak{g}) \underset{U(\mathbb{C} \cdot K)}{\otimes} \mathbb{C}_\kappa, \quad \kappa \in \mathbb{C},$$

where one denotes by $\mathbb{C}_\kappa$ the one dimensional $\mathbb{C}[K]$-module on which K acts by κ.

(a) $U_1(\mathfrak{g})$ is the algebra D_V of polynomial differential operators on V and $U_0(\mathfrak{g}) = \mathcal{O}(T^*V)$. Moreover, the family $U_\kappa(\mathfrak{g}) = D_V$, $\kappa \in \mathbb{C}$, is the standard deformation of $\mathcal{O}(T^*V)$ to D_V.

(b) Consider the associative algebra $D_V = \oplus_{I,J} \mathbb{C}\cdot x^I v^J$ (we choose dual bases v_i of V and x_i of V^*), as a Lie algebra. Then $V \oplus V^* \oplus \mathbb{C}\cdot 1 \subseteq D_V$ is a Lie subalgebra. This is the Heisenberg Lie algebra for the symplectic vector space T^*V.

(c) Algebra $U\mathfrak{g}$ consists of global sections of the bundle of algebras $U_\kappa\mathfrak{g}$ over $\mathbb{A}^1 = \text{Spec } \mathbb{C}[K]$.

Proof. (a) $U\mathfrak{g}$ is the algebra generated by $V, V^*, \mathbb{C}K$, relation $[v, v^*] = \langle v|v^*\rangle \cdot K$ for $v \in V, v^* \in V^*$ and the commutativity relations for other pairs of spaces. So, $U_\kappa\mathfrak{g}$ is generated by V, V^*, relation $[v, v^*] = \kappa \cdot \langle v|v^*\rangle$ and commutativity relations between v' and also between v^*'s. If we think of $\langle v|v^*\rangle$ as the derivative of the linear function v^* on V with respect to the invariant vector field v, these are precisely the relations in the deformation of $\mathcal{O}(T^*V)$ to D_V.

2.4. **Verma modules.**

2.4.1. *Induction.* For a Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ consider the forgetful functor $\mathcal{F}^{\mathfrak{g}}_{\mathfrak{b}}$: $Rep(\mathfrak{g}) \to Rep(\mathfrak{b})$. In the opposite direction one has

$$Ind^{\mathfrak{g}}_{\mathfrak{b}} : Rep(\mathfrak{b}) \to Rep(\mathfrak{g}), \quad Ind^{\mathfrak{g}}_{\mathfrak{b}}(W) \stackrel{\text{def}}{=} U\mathfrak{g} \underset{U\mathfrak{b}}{\otimes} W.$$

2.4.2. *Triangular Lie algebras.* We say that $\mathfrak{g}$ is triangular if it has a decomposition into a sum of vector spaces $\mathfrak{g} = \mathfrak{g}_{<0} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{>0}$ such that (0) summands are Lie subalgebras, (ii) $\mathfrak{g}_0$ is abelian, (iii) $[\mathfrak{g}_0, \mathfrak{g}_?] \subseteq \mathfrak{g}_?$ for $/ = < 0, > 0$.

Then $\mathfrak{g}_{\geq 0} \stackrel{\text{def}}{=} \mathfrak{g}_0 \oplus \mathfrak{g}_{>0}$ is a Lie subalgebra and $\mathfrak{g}_{\geq 0} \stackrel{p}{\to} \mathfrak{g}\psi/\mathfrak{g}_{>0} \cong \mathfrak{g}_0$ is a map of Lie algebras.

2.4.3. *Example:* $\mathfrak{gl}_n$. In $\mathfrak{g} = \mathfrak{gl}_n$ consider $\mathfrak{g}_{>0}$ = strictly upper triangular matrices, $\mathfrak{g}_{<0}$ = strictly lower triangular matrices, and $\mathfrak{g}_0$ = diagonal matrices.

This triangular structure is inherited by some subalgebras, say $\mathfrak{sl}_n$.

2.4.4. *Example: Heisenberg Lie algebra.* In $\mathfrak{g} = \widetilde{T^*V}$, let $\mathfrak{g}_{>0} = V$, $\mathfrak{g}_{<0} = V^*$ and $\mathfrak{g}_0 = C\cdot K$. Then a $\lambda \in \mathfrak{g}^*$ is the same as a number $\kappa = \langle \lambda, K\rangle \in \mathbb{C}$.

2.4.5. *Verma modules.* Verma module V_λ associated to a triangular Lie algebra $\mathfrak{g}$ and $\lambda \in (\mathfrak{g}_0)^*$ is the $\mathfrak{g}$-module

$$V_\lambda \stackrel{\text{def}}{=} Ind^{\mathfrak{g}}_{\mathfrak{g}\psi} \mathbb{C}_\lambda.$$

Here, $\mathbb{C}_\lambda$ is the one-dimensional vector space $\mathbb{C}$ which is considered as a module for $\mathfrak{g}_0$ by $\mathfrak{g}_0 \stackrel{\lambda}{\to} \mathbb{C} = \text{End}_{\mathbb{C}}(\mathbb{C})$. We can make it into a module for $\mathfrak{g}\psi$ via the Lie algebra map $\mathfrak{g}_{\geq 0} \stackrel{p}{\to} \mathfrak{g}_0$, and then we induce it to $\mathfrak{g}$.

2.4.6. *Lemma.* (a) Vector $|\lambda\rangle \stackrel{\text{def}}{=} 1_{U\mathfrak{g}} \otimes 1_{\mathbb{C}} \in V_\lambda$ (called the vacuum vector), is killed by $\mathfrak{g}_{>0}$ and $\mathfrak{g}_0$ acts on it by λ.

(b) There is a canonical identification of $U\mathfrak{g}_{<0}$-modules $V_\kappa \cong U(\mathfrak{g}_{<0})$, given by the action on the vacuum vector $U\mathfrak{g}_{<0} \ni u \mapsto u|\lambda\rangle \in V_\lambda$.

2.4.7. *Verma modules for the Heisenberg algebra.* Here $\mathfrak{g} = \widetilde{T}^*V$ is the triangular Lie algebra associated to the symplectic vector space $A = T^*V$.

2.4.8. *Lemma.* (a) The action of $U\mathfrak{g}$ on V_κ factors to an action of $U_\kappa\mathfrak{g}$.

(b) At $\kappa = 1$, the action of $U_1\mathfrak{g}$ on V_1 is the same as the action of D_V on $\mathcal{O}(V)$.

(c) At $\kappa = 0$, the action of $U_0\mathfrak{g}$ on V_0 is the same as the action of $\mathcal{O}(T^*V)$ on $\mathcal{O}(V)$.

Proof. (a) K act by κ on the vacuum vector. Since K is in the center of $U\mathfrak{g}$, it acts by κ on $U\mathfrak{g}_{<0} \cdot |\kappa\rangle = V_\kappa$.

(b) Recall that $U_1\mathfrak{g}$ is identified with D_V so that the generators $v \in V$ and $v^* \in V^*$ are interpreted as a vector field on V and as a linear function on V. As a vector space V_k is identified with $U\mathfrak{g}_{<0} = U(V^*) = S(V^*) = \mathcal{O}(V)$, moreover, this is an identification of modules for $U\mathfrak{g}_{<0} = \mathcal{O}(V)$. The action of $v \in V$ on $V_1 = \mathcal{O}(V)$ kills $1 \in \mathcal{O}(V)$ (it corresponds to the vacuum vector and $V = \mathfrak{g}_{>0}$, hence the action on $f \in \mathcal{O}(V)$ is

$$v\ f = v \cdot (f \cdot 1) = (v \cdot f) \cdot 1 = [v, f] \cdot 1 + f \cdot v \cdot 1 = = v(f)$$

since v kills 1 and the bracket of v and f in D_V is the derivative $v(f)$.

2.5. **Affine Heisenberg Lie algebras and loops in a vector space.** We start with a "metrized" vector space, i.e., from an orthogonal or a symplectic vector space $\mathfrak{h}, (-,-)$. In either case, we consider the space $\mathfrak{h}[t, t^{-1}] = Map(G_m, \mathfrak{h})$ of "polynomial loops in $\mathfrak{h}$". This space has already appeared as the space $\Lambda_{pol}(\mathfrak{h})$ of polynomial maps from S^1 to $\mathfrak{h}$.

The metrized structure on $\mathfrak{h}$ gives a skew-symmetric bilinear form ω on $\mathfrak{h}[t, t^{-1}]$, and we get a Heisenberg Lie algebra $\widehat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C} \cdot K$.

Applications to strings will use the orthogonal version, i.e., bilinear form $(-,-)$ will be symmetric. Then one of the Verma modules V_1 for a subalgebra $\mathfrak{h}'$ of $\widehat{\mathfrak{h}}$ will be the quantization $\mathcal{H}_1$ of the space of based loops $\Omega(M)$ for a real form M of $\mathfrak{h}$. (In this case Verma module V_1 is irreducible, so $V_1 = L_1$ in the notation above.)

The infinitesimal version. Instead of loops $\mathfrak{h}[t, t^{-1}] = Map(G_m, \mathfrak{h})$ consider the infinitesimal loops $\mathfrak{h}((t)) = Map(\widehat{0} - 0, \mathfrak{h})$, where, $\widehat{0} =$ Spec $\mathbb{C}[[t]]$ is the formal disc (an ind-scheme) and $\widehat{0} - 0 =$ Spec $\mathbb{C}((t))$ is the punctured formal disc (a scheme over an ind-scheme).

2.5.1. *The symplectic case.* Let $\mathfrak{h}$ be a super vector space with a skew-symmetric form $(-,-)$. It gives a skew-symmetric form ω on $\mathfrak{h}[t,t^{-1}]$ by $\omega(f,g)\stackrel{\text{def}}{=} Res(f,g)$. This makes $\widehat{\mathfrak{h}}\stackrel{\text{def}}{=}\mathfrak{h}[t,t^{-1}]\oplus\mathbb{C}K$ into a Lie algebra with K central and

$$[f,g]= Res(f,g)\cdot K, \quad f,g\in\mathfrak{h}[t,t^{-1}].$$

This is the most obvious construction, however it is not quite natural since we need to choose a local parameter t to define the residue of a function.

For $x\in\mathfrak{h}$ denote $x_n= xt^n,\ n\in\mathbb{Z}$. In these coordinates the bracket is

$$[x_m,y_n]= Res(m\cdot xt^{m-1}dt, yt^n)\cdot K= \delta_{m,-n}\ m\cdot(x,y)\cdot K.$$

The triangular structure is given by $\widehat{\mathfrak{h}}_{>0}= \widehat{\mathfrak{h}}_+\oplus\mathfrak{h}_-$ where $\widehat{\mathfrak{h}}_{>0}\stackrel{\text{def}}{=} t\mathfrak{h}[t],\ \widehat{\mathfrak{h}}_{<0}\stackrel{\text{def}}{=} t^{-1}\mathfrak{h}[t^{-1}]$ and $\widehat{\mathfrak{h}}_0= \mathfrak{h}\oplus\mathbb{C}\cdot K$.

2.5.2. *The quadratic case.* Let $\mathfrak{h}$ be a (super) vector space with a symmetric form $Q=(-,-)$. It gives a skew-symmetric form on $\mathfrak{h}[t,t^{-1}]$ by $\omega(f,g)\stackrel{\text{def}}{=} Res(df,g)$. This makes $\widehat{\mathfrak{h}}\stackrel{\text{def}}{=}\mathfrak{h}((t))\oplus\mathbb{C}K$ into a Lie algebra with K central and

$$[f,g]\stackrel{\text{def}}{=} Res(df,g)\cdot K, \quad f,g\in\mathfrak{h}((t)).$$

Here the derivative is used to make the expression skew-symmetric. The construction is canonical since the residue is applied to 1-forms.

The triangular structure is given by $\widehat{\mathfrak{h}}_{>0}= \widehat{\mathfrak{h}}_+\oplus\mathfrak{h}_-$ where $\widehat{\mathfrak{h}}_{>0}\stackrel{\text{def}}{=} t\mathfrak{h}[[t]],\ \widehat{\mathfrak{h}}_{<0}\stackrel{\text{def}}{=} t^{-1}\mathfrak{h}[t^{-1}]$ and $\widehat{\mathfrak{h}}_0= \mathfrak{h}\oplus\mathbb{C}\cdot K$.

2.5.3. *Lie algebra decomposition* $\widehat{\mathfrak{h}}=\mathfrak{h}'\oplus\mathfrak{h}$. Let $\mathfrak{h}'\stackrel{\text{def}}{=} L^0(\mathfrak{h})\oplus\mathbb{C}\cdot K)\subseteq\widehat{\mathfrak{h}}$.

2.5.4. *Lemma.* (a) $\mathfrak{h}((t))= L^0(\mathfrak{h})\oplus\mathfrak{h}$ with the first summand symplectic for ω and $\mathfrak{h}\perp\mathfrak{h}((t))$ for ω. (This decomposition is a characteristic feature of the quadratic case which uses the derivative in the definition of the form.)

(b) This leads to a decomposition into a sum of Lie algebras $\widehat{\mathfrak{h}}=\mathfrak{h}'\oplus\mathfrak{h}$ with $\mathfrak{h}$ abelian.

(c) $(L^0(\mathfrak{h}),\omega)$ is the cotangent bundle to $J^0(\mathfrak{h})$.

For the induced triangular structure on $\mathfrak{h}'$ one has $\mathfrak{h}'_0= \mathbb{C}\cdot K$ and $\mathfrak{h}'_{>0}=\widehat{\mathfrak{h}}_{>0},\ \mathfrak{h}'_{<0}=\widehat{\mathfrak{h}}_{<0}$.

2.6. **$U_1\mathfrak{h}'$ is an algebra of differential operators.** Here we will suppose that the bilinear form is non-degenerate.

2.6.1. *Geometric language.* One can think of the space $\mathfrak{h}[t]$ as polynomial (algebraic) maps from the line $\mathbb{A}^1 = \mathbb{C}$ to $\mathfrak{h}$ - for $\mathfrak{h} = \mathbb{C}$ the space of maps from $\mathbb{A}^1$ to $\mathfrak{h}$ would be the space of polynomial functions on $\mathbb{A}^1$, i.e., $\mathbb{C}[t]$. It lies in the space $\mathfrak{h}[t, t^{-1}]$ of maps from $G_m = \mathbb{A}^1 - 0$ to $\mathfrak{h}$ (since $\mathcal{O}(G_m) = \mathbb{C}[t, t^{-1}]$).

Let $D \stackrel{\text{def}}{=}$ Spec $\mathbb{C}[[t]]$ be the infinitesimal disc (an ind-scheme) and $D^* \stackrel{\text{def}}{=}$ Spec $\mathbb{C}((t))$ the infinitesimal punctured disc (a scheme over an ind-scheme).

We will consider the maps of D into $\mathfrak{h}$ (called jets or arcs) and maps from and D^* into $\mathfrak{h}$ (called loops),

$$J(\mathfrak{h}) \stackrel{\text{def}}{=} Map(D, \mathfrak{h}) = \mathfrak{h}[[t]] \subseteq L(\mathfrak{h}) \stackrel{\text{def}}{=} Map(D^*, \mathfrak{h}) = \mathfrak{h}((t)),$$

and also the spaces of based jets and loops

$$J^0(\mathfrak{h}) \stackrel{\text{def}}{=} \{\gamma : D \to \mathfrak{h},\ \gamma(0) = 0\} = t \cdot \mathfrak{h}[[t]] \subseteq L^0(\mathfrak{h}) \stackrel{\text{def}}{=} \{\gamma = \sum \gamma_n t^n \in \mathfrak{h}((t)),\ \gamma_0 = 0\}.$$

2.6.2. *Lemma.* (a) $U_1\mathfrak{h}'$ is dense in the ring of differential operators on based jets $D_{t\mathfrak{h}[[t]]} = D_{J^0(\mathfrak{h})}$.

(b) $U_0\mathfrak{h}'$ is dense in the ring of functions $\mathcal{O}(L^0(\mathfrak{h}))$ on based loops.

2.6.3. *Remarks.* (1) The continuous modules are the same for differential operators and for the affine Heisenberg Lie algebra $\mathfrak{h}'$ at $K = 1$.

(2) I will use here the infinitesimal loops $\widehat{\mathfrak{h}} = \mathfrak{h}((t)) \oplus \mathbb{C}K$. However, the results are the same for the smaller algebras of polynomial loops.

Proof. This is just a case of the claim for arbitrary Heisenberg Lie algebras lemma 2.3.2 but we write the details anyway.

The positive part $\mathfrak{h}'_{>0} = t\mathfrak{h}[[t]]$ is literally the vector space $J^0(\mathfrak{h})$, so we can think of it as (translation invariant) vector fields on $J^0(\mathfrak{h})$.

Next, we will identify the negative part $\mathfrak{h}'_- = \oplus_{i<0} t^i\mathfrak{h}$ with the linear functions on a topological vector space $J^0(\mathfrak{h}) = t\mathfrak{h}[[t]]$ via the pairing $\omega(f, j) \stackrel{\text{def}}{=} Res(df, j)$ of $f \in \mathfrak{h}'_-$ and $j \in t\mathfrak{h}[[t]]$. For $k, l > 0$ and $b, c \in \mathfrak{h}$, the pairing is $\omega(bt^{-k}, ct^l) = \delta_{k,l}\ k \cdot (b, c)$. So bt^{-k} becomes a linear function $k(b, -)$ on the copy $\mathfrak{h}t^k \subseteq t\mathfrak{h}[[t]]$ of $\mathfrak{h}$.

For $i, j > 0$, and $a, b \in \mathfrak{h}$, the bracket in $U_1\mathfrak{h}'$ is $[at^i, bt^{-j-1}] = Res(d(at^i), bt^{-j-1}) = \delta_{i,j}\ (i \cdot a, b)$, the same as the bracket of a vector field at^i on $t\mathfrak{h}[[t]]$, and a linear function $j(b, -)$ which is extended from $\mathfrak{h}t^j \cong \mathfrak{h}$ to $t\mathfrak{h}[[t]]$ by 0.

(b) The commutator in $U_k\widehat{\mathfrak{h}}$ is $[at^i, bt^{-j-1}] = k \cdot \delta_{i,j}\ (i \cdot a, b)$. So $U_0\mathfrak{h}' = U(\mathfrak{h}'/\mathbb{C}K) = U[L^0(\mathfrak{h})]$ is the commutative algebra $S[L^0(\mathfrak{h})]$. Since $L^0(\mathfrak{h})$ is a self-dual vector space for the pairing $(f, g) \stackrel{\text{def}}{=} Res(df, g)$ as above, we can think of $S[L^0(\mathfrak{h})]$ as polynomial functions on the vector space $L^0(\mathfrak{h})$.

2.6.4. *Deformation.* We say that the specialization $U_k\mathfrak{h}'$ where K becomes a number k is the specialization at level k. Family of algebras $U_k\mathfrak{h}'$ is a deformation of functions on based loops to differential operators on based jets.

Where does this deformation come from? Mathematically, there is one miraculous fact: based loops in a vector space are the cotangent bundle to the based jets.

2.6.5. *Remarks.* (0) The above notion of "based loops" as a subset of loops does not really make sense since it requires a choice of a local parameter t. The natural space is the quotient of loops modulo the constant loops.

(1) The whole story works the same for smooth loops into a real vector space with an inner product

$$\omega(f,g) = \int_{S^1} (df, g).$$

2.6.6. *Verma modules for* $\mathfrak{h}'$. We will just specialize the claims of lemma 2.4.8. Verma modules for $\mathfrak{h}'$ are parameterized by linear functionals on $\mathfrak{h}'_0 = \mathbb{C}\cdot K$, i.e., by numbers k. Observe that the action of $U\mathfrak{h}'$ on V_k factors to an action of $U_k\mathfrak{h}'$.

With the arguments from the lemma we see that the $U_1\mathfrak{h}'$-module V_1 can be identified with the $D_{J^0(h)}$-module $\mathcal{O}(J^0(h))$, while the action of $U_0\mathfrak{h}'$ on V_0 can be identified with the action of the ring of functions on $L^0(h)$, on the vector space of functions on $J^0(\mathfrak{h}) \subseteq L^0(h)$.

2.7. **Affine Lie algebras and loops in a compact Lie group.** Starting from a compact real Lie group U we get a complex Lie group $G = U_{\mathbb{C}}$. Its Lie algebra $\mathfrak{g}$ has a triangular structure, for instance for $U = (n)$ one has $G = GL_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}_n$ decomposes into $\mathfrak{g}_{>0}$ = strictly upper triangular matrices, $\mathfrak{g}_{<0}$ = strictly lower triangular matrices, and $\mathfrak{g}_0$ = diagonal matrices.

This gives a triangular structure of $\widehat{\mathfrak{g}}$ with $\widehat{\mathfrak{g}}_{>0} = t\mathfrak{g}[t]\oplus\mathfrak{g}_{>0}$, $\widehat{\mathfrak{g}}_{<0} = t^{-1}\mathfrak{g}[t^{-1}]\oplus\mathfrak{g}_{<0}$ and $\widehat{\mathfrak{g}}_0 = \mathfrak{g}_0\oplus\mathbb{C}\cdot K$.

Now we have Verma modules V_λ and their irreducible quotients L_λ. One of these is the quantization $\mathcal{H}_1$ of $\Omega(U)$ which has been constructed geometrically as holomorphic sections of a line bundle $\mathcal{L}$ over $\Omega(U)$.

2.8. **Extras.**

2.8.1. *Vertex algebras.* Verma modules V_k come with the additional structure of a vertex algebra.

The underlying vector space of modules V_k is $U(\mathfrak{h}'_{<0}) = \mathcal{O}(t\mathfrak{h}[[t]])$, it is called the Fock space. It also has an additional structure of a (co)commutative Hopf algebra - the space of functions on a vector space.

2.8.2. *Polarizations.* In the quadratic case the metric on $\mathfrak{h}$ has two standard forms and these lead to two interpretations of $U_1\widehat{\mathfrak{h}}$.
(i) The diagonal form $Q = \sum x_i^2$ hence $\mathfrak{h} = \oplus\, \mathfrak{h}_i$ is compatible with the above interpretation in terms of differential operators on the "disc space" $\mathfrak{h}^*[[t]]$.
(ii) The polarized form $Q = \sum x_i y_i$ comes with a a polarization of $\mathfrak{h}$: $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, with $\mathfrak{h}_\pm$ maximal isotropic and dual to each other via Q. It leads to an interpretation by differential operators on the loop space $\mathfrak{a}^*((t))$.

2.8.3. *Extension to arbitrary Riemannian manifolds.* The above story with cylinders in Minkowski space $\mathfrak{h}_\mathbb{R}$, may work for a general Riemannian manifold M. The tangent space at a loop γ consists of sections of $\gamma^* T^M$ and we want to use $\omega(f,g) = \int_{S^1} (df, g)$, for that one needs to make sense of the action of $\frac{\partial}{\partial\theta}$ on sections of $\gamma^* T^M$. This is the Levi-Civita connection for the pull-back $\gamma^* G$ of the metric G on M. (See Brylinski's book.)

The extension when M is a compact Lie group U seems "known". In this case the symplectic structure on based loops is familiar as either the Kähler structure on the loop Grassmannian $\mathcal{G}$ or as the central extension cocycle [Pressley-Segal].

In that case based smooth loops may be interpreted as $\Lambda(U) = = \{f \in C^\infty(S^1, U),\ f(1_{S^1}) = 1_U\}$. There are two quotient interpretations of this space $C^\infty(S^1, U)/U = \Lambda(U) = U\backslash C^\infty(S^1, U)$, and each leads to a Kähler structure on $\Lambda(U)$. The corresponding real symplectic structures are generalizations of the above story. Now, the Hilbert space of the quantization of loops is the Verma module or its irreducible quotient (in this case it may be the basic representation of the loop group, i.e., the sections of the determinant line bundle). We can think of it as the quantization of a Kähler structure on the loop Grassmannian. (So there would be no Lagrangian subspace, only a complex polarization of the real manifold $\Lambda(U)$.

Physicists have already considered such constructions. In order to be in the critical dimension one takes maybe a semi-direct product of a group with a representation.

Part VII. Super-symmetric quantum mechanics

0. Intro

0.0.4. *The ingredients.* We will combine the bosonic free particle $L = \frac{1}{2}\dot{x}^2$ with two fermionic free particles $L = \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2$ which we treat as a "complex fermionic free particle" $L = i\bar{\psi}\dot{\psi}$.

One can also add a Morse function W which appears in Witten's formulation of Morse theory as a Lagrangian term $\frac{1}{2}(W')^2 + \frac{W''}{2}(\bar{\psi}\psi - \psi\bar{\psi})$.

In the local case, related to the manifold $M = \mathbb{A}^1 = \mathbb{R}$ with the standard metric we will call this QM. In the "global case" of a manifold (M, G) we talk of a σ-model.

In the global case this action has no SUSY and we need to add a curvature term $\mathcal{C}_{\psi,\bar{\psi}}\psi \underset{G}{\cdot} \bar{\psi}$ to restore the supersymmetry.

0.0.5. We will use Euclidean Lagrangian (so that one can compute the path integrals) and the Minkowski Hamiltonian (so that we follow physics).

1. 0-dimensional QFT = integrals

0. Intro

0.1. **Super linear algebra.** A super vector space is simply a vector space graded by $\mathbb{Z}_2 = \{0, 1\}$: $V = V_0 \oplus V_1$. We will say that vectors $v \in V_\psi$ are homogeneous of parity p and we will denote the parity by $\bar{v} \overset{\text{def}}{=} p$. Another notation: fermionic sign $(-1)^F$ is the linear operator which is $+1$ on V_0 and -1 on V_1.

0.1.1. *Algebra.* A super algebra is an algebra A with a compatible super structure, i.e., $A_\psi \cdot A_q \subseteq A_{p+q}$. In the presence of the super-structure one can use super-versions of standard ideas, for instance
(1) super-dimension (fermionic dimension) is $\dim_F(V) = \dim(V_0) - \dim(V_1)$,
(2) super-trace of a linear operator $T : \mathbb{V} \to V$ is defined using its block decomposition $\mathrm{tr}_F(T) = \mathrm{tr}(T_{00}) - \mathrm{tr}(T_{11})$,
(3) super-commutator in a super-algebra A is $[a, b]_F = \overset{\text{def}}{=} ab - (-1)^{\bar{a}\bar{b}}ba$.
This is an example of the main calculational rule:

"The *Rule*: when a passes b a sign $(-1)^{\bar{a}\bar{b}}$ appears.

(4) elements a and b of a super-algebra A super-commute if $ab = (-1)^{\bar{a}\bar{b}}ba$.
The notion of super-commutativity: (i) allows one to think of some non-commutative situations as if they were commutative, (ii) gives a notion of a super-commutative algebra, i.e., of a super-space.

0.2. **Super-manifolds.** Super versions $\mathbb{A}^{n|m}$ of affine spaces are given by the super-commutative algebras of functions on these spaces

$$\mathcal{O}(\mathbb{A}^{n|m}) = \mathbb{R}[x^1, ..., x^n, \psi^1, ..., \psi^m] = S^*(\mathbb{R}x^1 \oplus \cdots \oplus \mathbb{R}x^n) \otimes \overset{*}{\wedge} (\mathbb{R}\psi^1 \oplus \cdots \oplus \mathbb{R}\psi^m),$$

with x^μ's even and ψ^k's odd. It is a product of the affine space $\mathbb{A}^n = \mathbb{A}^{n|0}$ and a super point $\mathbb{A}^{0|m}$. A basis of $\mathcal{O}(\mathbb{A}^{0|m}$ is given by the monomials $\psi^{i_1<\cdots<i_k} \overset{\text{def}}{=} \psi^{i_1} \cdots \psi_{i_k}$.

0.2.1. *Differentiation.* A vector field means a derivative of the algebra of functions, so the vector fields on $\mathbb{A}^{n|m}$ are all $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} + \xi^k \frac{\partial}{\partial \psi^k}$, where $\frac{\partial}{\partial \psi^k}$ has the usual properties that it kills x^μ's and $\frac{\partial}{\partial \psi^k} p^j = \delta_{jk}$, but

$$\frac{\partial}{\partial \psi^k}(fg) = \frac{\partial}{\partial \psi^k} f \; g + (-1)^{\bar{f}} f \; \frac{\partial}{\partial \psi^k} g.$$

This fits into the *Rule* when we say that $\frac{\partial}{\partial \psi^k}$ has odd parity.

The differential $df = \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial \psi^k} d\psi^k$ is of parity 0 so it satisfies $d(fg) = df \cdot g + f \cdot dg$.

The contraction of vector fields and 1-forms (differentials) $\langle \xi, \omega \rangle$ satisfies $\langle \xi, df \rangle = \xi(f)$ and the Parity Rule.

0.3. **Integration.** The integrals of functions on super affine spaces will be (ordinary) numbers.

0.3.1. *Super points.* Integration of functions on a super point $\mathbb{A}^{0|m}$ is defined by using successively the formula $\int d\psi \; a + b\psi \overset{\text{def}}{=} b.$, i.e., the integral just takes the highest degree coefficient (with a correct sign)

$$\int d\psi^n \cdots d\psi^1 \sum_{I = \{i_1 < \cdots < i_k\}} c_I \cdot \psi^I \overset{\text{def}}{=} c_{12\cdots m}.$$

(Analogy: for an analytic function $f(z)$ the integral over a circle $\int_{S^1} f(z) \; dz$ is the coefficient of z^{-1} in the Laurent power series expansion.)

0.3.2. *Super affine spaces.* Integrals on $\mathbb{A}^{n|m}$ are evaluated so that one first integrates over the fermionic variables and then we are left with an ordinary integral. For example if $S[x, \psi^1, \psi^2] = U(x) + V(x)\psi^1\psi^2$ then

$$\int_{\mathbb{A}^{1|2}} dx \; d\psi^2 \; d\psi^1 \; e^{-S[x,\psi^1,\psi^2]} = \int_{\mathbb{A}^{1|2}} dx \; d\psi^2 \; d\psi^1 \; e^{-U(x)} \sum_k \frac{(-1)^k}{k!} V(x)^k (\psi^1\psi^2)^k$$

$$= -\int_{\mathbb{A}^{1|0}} dx \; e^{-U(x)} \int_{\mathbb{A}^{0|2}} d\psi^2 \; d\psi^1 \; V(x)\psi^1\psi^2 = -\int_{\mathbb{A}^{1|0}} dx \; V(x) e^{-U(x)}.$$

0.4. **Gaussian integrals.** Let our 0-dimensional manifold Σ be a point. Maps from Σ to M are now elements of M and a path integral $\int_{x:|Si\to M} Dx\ e^{-S[x,g]}$ is an ordinary integral $\int_M Dx\ e^{-S[x,g]}$. Let M be a vector space and let the action S be quadratic in x, $S[x] = x^i A_{ij} x^j$ for a symmetric matrix A, then this is a Gaussian integral:

$$\int_M Dx\ e^{-\frac{1}{2}(x^i A_{ij} x^j)} = (\det M)^{\frac{1}{2}}$$

for $Dx = \prod \frac{dx^i}{\sqrt{2\pi}}$.

0.4.1. *Fermionic Gaussian integrals.* Now let M be a super point $(0, m)$. Quadratic functions on M are function of the form $S[x] = \psi^i B_{ij} \psi^j$ for an anti-symmetric B (so $\frac{1}{2}S[x] = \sum_{i<j} \psi^i B_{ij} \psi^j$). A fermionic Gaussian integral is

$$\int_M d\psi^m \cdots d\psi^1\ e^{-\frac{1}{2}\ \psi^i B_{ij} \psi^j} = \int_M d\psi^m \cdots d\psi^1\ e^{-\sum_{i<j}\ \psi^i B_{ij} \psi^j}$$

$$= \int_M d\psi^m \cdots d\psi^1 \sum_k \frac{(-1)^k}{k!} (\sum_{i<j} B_{ij} \psi^i \psi^j)^k.$$

Since we get only the even degree terms, this is zero if m is odd. If m is even this is

$$\int_M d\psi^m \cdots d\psi^1 \frac{(-1)^{m/2}}{(m/2)!} (\sum_{i<j} B_{ij} \psi^i \psi^j)^{m/2},$$

and we get a contribution $\frac{(-1)^{m/2}}{(m/2)!} B_{i_1 j_1} \cdots B_{i_{m/2} j_{m/2}}$, whenever $i_1, j_1 ..., i_{m/2}, j_{m/2}$ is an ordering of $1, ..., m$ such that $i_k < j_k$. Therefore, the Gaussian integral is the Pffafian of the "antisymmetric" quadratic form

$$\int_M d\psi^m \cdots d\psi^1\ e^{-\frac{1}{2}\ \psi^i B_{ij} \psi^j} = \frac{(-1)^{m/2}}{(m/2)!} \cdot Pf(B).$$

(The Pfafian of an antisymmetric matrix is a square root of its determinant $Pf(B)^2 = \det(B)$.)

0.5. **Super-symmetry of the action.** An ordinary (continuous) symmetry of a function S on M is a flow Φ on M (i.e., an action of $(\mathbb{R}, +)$ on M), which preserves S: $S(\Phi_r(x)) = S(x)$. An infinitesimal symmetry is a vector field ξ on M which preserves S in the sense that $\xi S = \langle dS, \xi \rangle$ vanishes. The same applies to super-spaces. In this setting the action will be an even function on a super manifold.

We say that the symmetry is bosonic or fermionic if the vector field is even or odd. For instance if $S(x, \psi^1, \psi^2)$ is independent of ψ^2 this is a fermionic symmetry – the vector field is $\frac{\partial}{\partial \psi^2}$ and the flow is the motion in the direction of ψ^2. (Observe that in that case the integral $\int_{\mathbb{A}^{1|2}} dx\ d\psi^2\ d\psi^1\ e^{-S[x,\psi^1,\psi^2]}$ vanishes - this is the source of simplicity in super-integrals.)

A super-symmetry (SUSY) is a symmetry which mixes even and odd, i.e., the vector field is neither even nor odd.

For an example, we will calculate the path integral for the action $S[x,\psi_1,\psi_2] = U(x) + V(x)\psi_1\psi_2$ when $U = \frac{1}{2}P^2$ and $V = -P'$ for a polynomial $P = P(x)$ on $\mathbb{A}^1$, and then we will see that the result can be explained using a SUSY of this action.

0.5.1. *Integration on* $\mathbb{A}^{1|0} = |R$. We will reduce the integral for $S[x,\psi_1,\psi_2] = \frac{1}{2}P(x)^2 - P'(x)\psi_1\psi_2$ to integration on the real line and the standard calculus. First, by 0.3.2

$$\int_{\mathbb{A}^{1|2}} Dx\ d\psi^2\ d\psi^1\ e^{-S} = \int_{\mathbb{R}} Dx\ P'(x)e^{-\frac{1}{2}P^2},$$

now substitution $u = P(x)$ gives a Gaussian integral

$$\int_{P(-\infty)}^{P(\infty)} Dx\ e^{-\frac{1}{2}u^2} = \deg(\widetilde{P}).$$

Here $\widetilde{P}$ is P interpreted as a map from S^1 to itself – this can be done since $P(\pm\infty)$ is one of $\{\pm\infty\}$. The degree of $\widetilde{P}$ is defined as the number of times $\widetilde{P}$ winds up the circle onto itself in positive direction. Observe that if $\deg P$ is even then $\deg(\widetilde{P})$ is 0 and this is also the value of the integral since $P(-\infty) = P(\infty)$. If $\deg P$ is odd then $\deg(\widetilde{P})$ is the sign $\varepsilon = \pm 1$ of the highest coefficient and this is also the value of the integral since $P(\pm\infty) = \varepsilon\cdot\pm\infty$.

0.5.2. *SUSY.* The vector field

$$\delta = (\psi_1+\psi_2)\frac{\partial}{\partial x} + P(x)\frac{\partial}{\partial\psi_1} - P(x)\frac{\partial}{\partial\psi_2}$$

is a supersymmetry of the action

$$\delta\, S = (\psi_1+\psi_2)(P(x)P'(x)-P''(x)\psi_1\psi_2) - P(x)\,P'(x)\,\psi_2 - (-P(x))\,P'(x)\frac{\partial}{\partial\psi_2}\psi_1\psi_2 = 0.$$

We check the sign in the last term: $\frac{\partial}{\partial\psi_2}\psi_1\psi_2 = -\psi_1\frac{\partial}{\partial\psi_2}\psi_2 = -\psi_1$.

Moreover, this is a SUSY of the measure $dx\ d\psi_2\ d\psi_1$ (a vector field acts on 1-forms by $\delta(df) = d(\delta f)$)

$$\delta\ (dx\ d\psi_2\ d\psi_1) = \delta(dx)\ d\psi_2\ d\psi_1 + dx\ \delta(d\psi_2)\ d\psi_1 + dx\ d\psi_2\ \delta(d\psi_1)$$

$$= d(\psi_1+\psi_2)\ d\psi_2\ d\psi_1 + dx\ dP(x)\ d\psi_1 + dx\ d\psi_2\ d(-P(x)) = 0$$

since in the last two terms one has dx twice and in the first term one of $d\psi_i$'s.

0.5.3. *Local arguments in* $\mathbb{A}^{1|2}$. *The first ingredient* is the vanishing of the integral – locally in $\mathbb{A}^{1|2}$ and as long as we stay away from the zeros of $P(x)$. This comes from supersymmetry.

Let $a \in A^1$, if $P(a) \neq 0$ then the odd component of $\delta(a)$ is non-trivial. Then one can change the coordinates near a to $\widetilde{x}, \widetilde{\psi}_1, \widetilde{\psi}_2$ so that $\delta = \frac{\partial}{\partial \psi_2}$ and the measure does not change: $d\widetilde{x}\ d\widetilde{\psi}_1\ d\widetilde{\psi}_2 = dx\ d\psi_1\ d\psi_2$ (i.e.the Jacobian is 1).

Notice that we can not have $\widetilde{x}$ even and $\widetilde{\psi}_i$ odd since δ is not odd. Still, one should be able to show that it implies the vanishing of the integral in the slices $\widetilde{x} = c$ of this neighborhood of a in $\mathbb{A}^{1|2}$.

The second ingredient is the special case $P = x$. Then the integral is the Gaussian integral

$$\int_{\mathbb{R}} Dx\ P'(x)e^{-\frac{1}{2}P^2} = \int_{\mathbb{R}} Dx\ \alpha e^{-\frac{1}{2}\alpha^2 x^2} = \frac{\alpha}{\sqrt{\alpha}} = sign(\alpha) = sign(P'(b)).$$

0.5.4. *Conclusion.* The first step reduces the calculation to neighborhoods of zeros b of $P(x)$. There $P(x)$ is approximated by a line $\alpha(x-b)$. However, since $P = \alpha(x-b)$ has no other zeros, by the first step the integral near b is the global integral.

So the total integral is the sum of signs of $P'(x)$ at zeros of $P(x)$, i.e., we count $+1$ when P crosses the x-axis upwards and -1 when P crosses the x-axis downwards. The sum of these is the degree of the extension $\widetilde{P}$ of P to S^1.

0.5.5. *Remarks.* (1) One should still describe how the local contributions glue together. For instance, over an interval $\mathcal{I} \subseteq \mathbb{R}$, where P does not vanish, integral is usually not zero in the inverse $\{x \in \mathcal{I}\}$ of $\mathcal{I}$ in $\mathbb{A}^{1|2}$, though it vanishes in the "slanted inverse" $\{\widetilde{x} \in \mathcal{I}\}$.) So the integral can be large in a "small" difference of these spaces.

(2) Physicists would describe a vector field by $\delta x = \varepsilon(\psi_1 + \psi_2)$, $\delta\psi_1 = \varepsilon P(x)$, $\delta\psi_2 = -\varepsilon P(x)$ where ε is an infinitesimal (i.e., $\varepsilon^2 = 0$) real number.

1. 1d quantum field theory: quantum mechanics

1.0.6. *Euclidean Lagrangian.* On $M = \mathbb{A}^{1|2}$ we consider the Lagrangian

$$L = L_E = \frac{1}{2}[\dot{x}^2 + (W')^2] + i\bar{\psi}\dot{\psi} + \frac{W''}{2}(\bar{\psi}\psi - \psi\bar{\psi}).$$

Here we use the complexified fermions $\psi = \psi_1 + i\psi_2$ and $\bar{\psi} = \psi_1 - i\psi_2$.

1.0.7. *Remarks.* (1) One has, $i\bar{\psi}\psi = i(\psi_1 - i\psi_2)(\psi_1 + i\psi_2) = -2\psi_1\psi_2$ and

$$\bar{\psi}\dot{\psi} = (\psi_1 - i\psi_2)(\dot{\psi}_1 + i\dot{\psi}_2) = \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 + i(\psi_1\dot{\psi}_2 - \psi_2\dot{\psi}_1) = \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 + i(\psi_1\dot{\psi}_2 + \dot{\psi}_1\psi_2)$$
$$= \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 + i(\psi_1\psi_2)^{\cdot}.$$

Also,

$$\frac{1}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) = \bar\psi\dot\psi - \frac{1}{2}(\bar\psi\psi)^{\cdot} = \psi_1\dot\psi_1 + \psi_2\dot\psi_2 + i(\psi_1\psi_2)^{\cdot} - \frac{1}{2}(2i\psi_1\psi_2)^{\cdot} = \psi_1\dot\psi_1 + \psi_2\dot\psi_2.$$

(2) Instead of simply writing $\bar\psi\psi$, one often writes $\frac{1}{2}(\bar\psi\psi - \psi\bar\psi)$. These are classically the same but the second formula is correct on the quantum level – it is the symmetric quantization of the first.

(3) On often allows changing Lagrangian up to a total derivative. This affects the action only through the boundary terms, so it does not affect the path integral when the path is prescribed on the boundary. In this sense, $\bar\psi\dot\psi$ is the same as $\psi_1\dot\psi_1 + \psi_2\dot\psi_2$ or $\frac{1}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi)$.

1.0.8. *The variation.*

$$\delta L_E = \dot x\,\delta\dot x + W'W''\,\delta x + i(\delta\bar\psi\,\dot\psi + \bar\psi\,\delta\dot\psi) + \frac{W'''}{2}\delta x(\bar\psi\psi - \psi\bar\psi) + \frac{W''}{2}(\delta\bar\psi\,\psi + \bar\psi\,\delta\psi - \delta\psi\,\bar\psi - \psi\,\delta\bar\psi).$$

1.1. **Canonical formalism.** We start with the conjugate momenta $x \mapsto p \overset{\text{def}}{=} \frac{\partial L}{\partial \dot x} = \dot x$ and $\psi \mapsto \Pi \overset{\text{def}}{=} \frac{\partial L}{\partial \dot\psi} = -i\bar\psi$. (Also, $\bar\psi \mapsto 0$.)

1.1.1. If one replaces $i\bar\psi\dot\psi$ with $i\frac{1}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) = = i(\psi_1\dot\psi_1 + \psi_2\dot\psi_2)$ then $\psi \mapsto -i\frac{1}{2}\bar\psi$, $\bar\psi \mapsto -i\frac{1}{2}\psi$ and $\psi_k \mapsto ip_k$, $k = 1, 2$.

1.1.2. *Lemma.* (a) $\{p, x\} = 1$, $\{\bar\psi, \psi\} = -i$, while x, p commute with $\psi, \bar\psi$, and $\{\psi, \psi\} = 0 = \{\bar\psi, \bar\psi\}$.

(b) At $\hbar = 1$, the non-trivial commutators are

$$[x, p] = i, \quad [\psi, \bar\psi] = 1 = [\bar\psi, \psi],$$

while x, p commute with $\psi, \bar\psi$, and $0 = \psi^2 = \bar\psi^2$.

Proof. We use the following rule: the non-zero Poisson brackets between coordinates come from the conjugate momenta, i.e., $1 = \{p, q\}$ means that $1 = \{\frac{\partial L}{\partial \dot x}, x\}$. In the same way $1 = \{\frac{\partial L}{\partial \dot\psi}, \psi\} = \{\Pi, \psi\} = -i\{\bar\psi, \psi\}$. This quantizes to $[\widehat p, \widehat x] = \frac{\hbar}{i}1$ and $[\widehat{\bar\psi}, \widehat\psi] = \frac{\hbar}{i}i$. Notice that $\{\bar\psi, \psi\} = \{\psi, \bar\psi\} = i$ gives $[\bar\psi, \psi] = [\psi, \bar\psi] = \hbar$.

Finally, for any odd quantity ψ, relation $\{\psi, \psi\} = 0$ gives $0 = [\widehat\psi, \widehat\psi] = 2\widehat{\psi^2}$.

1.1.3. *Minkowski Hamiltonian.* We start with the Minkowski Lagrangian

$$L_M = \frac{1}{2}[\dot x^2 - (W')^2] + i\bar\psi\dot\psi - \frac{W''}{2}(\bar\psi\psi - \psi\bar\psi),$$

the Hamiltonian is then

$$H = H_M = p\dot x + \Pi\dot\psi - L_M = \frac{1}{2}[\dot x^2 + (W')^2] + \frac{W''}{2}(\bar\psi\psi - \psi\bar\psi).$$

1.2. **Symmetries of the action.** We will deal with "on shell" infinitesimal symmetries – vector fields which preserve the action *at the classical solutions only* ("off-shell" would mean a symmetry on the space of all fields). This can be remedied formally by adding another contribution $\frac{1}{2}z^2$ (for an even variable z), to the the action and extending the symmetries bellow (i.e., one needs to prescribe $\delta_i z$). The principle behind this is that one needs the same number of even and odd degrees of freedom to have SUSY (supersymmetry).

1.2.1. *Lemma.* (a) The Hamiltonian vector fields $\{-, Q_i\}$ given by the functions (called charges)

$$Q_1 = \frac{1}{2}p(\psi+\bar{\psi}) + \frac{i}{2}W'(\psi-\bar{\psi}), \quad Q_2 = \frac{1}{2}p(\psi-\bar{\psi}) - \frac{i}{2}W'(\psi+\bar{\psi})$$

are "on shell" infinitesimal symmetries of the action.

(b) The functions Q_i quantize to operators on the Hilbert space by the same formulas

$$Q_1 = \frac{1}{2}p(\psi+\bar{\psi}) + \frac{i}{2}W'(\psi-\bar{\psi}), \quad Q_2 = \frac{1}{2}p(\psi-\bar{\psi}) - \frac{i}{2}W'(\psi+\bar{\psi}).$$

These charges $\widehat{Q}_i$ commute with $\widehat{H}$.

Proof. (a) The vector fields $\delta_i = \{-, Q_i\}$ are given by

$$\delta_1 x = \frac{\varepsilon_1}{2}(\psi+\bar{\psi}), \quad \delta_1\psi = \frac{\varepsilon_1}{2}(i\dot{x}+W'), \quad \delta_1\bar{\psi} = \frac{\varepsilon_1}{2}(i\dot{x}-W'),$$

$$\delta_2 x = \frac{\varepsilon_2}{2}(\psi-\bar{\psi}), \quad \delta_2\psi = \frac{\varepsilon_2}{2}(\dot{x}-iW'), \quad \delta_2\bar{\psi} = \frac{\varepsilon_2}{2}(-\dot{x}-iW').$$

Now one plugs in these expressions into δL_E and obtains 0 if $x, \psi, \bar{\psi}$ is a classical solution.

(b) The operators Q_i satisfy
$i[Q_1, x] = \frac{1}{2}(\psi+\bar{\psi}), \quad i[Q_1,\psi] = \frac{1}{2}(i\dot{x}+W'), \quad i[Q_1,\bar{\psi}] = \frac{1}{2}(i\dot{x}-W'),$
and
$i[Q_2, x] = \frac{1}{2i}(\psi-\bar{\psi}), \quad i[Q_2,\psi] = \frac{1}{2}(\dot{x}-iW'), \quad i[Q_2,\bar{\psi}] = \frac{1}{2}(-\dot{x}-iW').$

(c) $[Q_i, Q_j] = \delta_{ij}\, H$, i.e., $H = 2{Q_i}^2$ and $Q_1Q_2 + Q_2Q_1 = 0$.

1.2.2. *Supercharges Q_i.* We see that these are the fundamental objects since the Hamiltonian can be reconstructed from supercharges. So at least in the operator approach, it does not matter that the Hamiltonian vector fields for functions Q_i are only "on shell" infinitesimal symmetries of the action since one has relations $[Q_i, Q_j] = \delta_{ij}\, H$.

1.2.3. *$N = 2$ SUSY.* This means that we have two super-symmetries Q_1, Q_2.

1.3. **The Hilbert space.** This is the tensor product of a bosonic and fermionic Hilbert spaces.

$$\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f = \mathbb{C}[\bar{a}] \otimes \mathbb{C}[\bar{\psi}] = \mathbb{C}[\bar{a}, \bar{\psi}], \qquad \text{and} \qquad \mathbb{C}[\bar{\psi}] = C \oplus \mathbb{C}\bar{\psi}.$$

The comparison of $\mathcal{H}_b$ and $\mathcal{H}_f$: $x^n \in \mathcal{H}_b$ is the state that consists of n bosonic particles, however, we can have at most one fermionic particle.

1.3.1. *The action of a super Heisenberg Lie algebra on $\mathcal{H}$.* Here p, x generate an action of a Heisenberg Lie algebra for the symplectic vector space T^*M_0 (for $M = \mathbb{R}$), and $\psi, \bar{\psi}$ do the same for the odd symplectic vector space T^*M_1 (for $M_1 = \mathbb{C}\psi$). In the case $W = 1$ of the Hamiltonian is a sum of an even harmonic oscillator $\frac{1}{2}[\dot{x}^2 + x^2]$ and an odd one $\bar{\psi}\psi = \frac{1}{2}(\bar{\psi}\psi - \psi\bar{\psi})$.

The action on $\mathcal{H}_f = \mathbb{C}|0\rangle \oplus \bar{\psi}\mathbb{C}|0\rangle$ is by $\psi\ |0\rangle = 0$ and $\psi\ \bar{\psi}|0\rangle = (\psi\ \bar{\psi} + \bar{\psi}\psi - \bar{\psi}\psi)|0\rangle = 1 \cdot |0\rangle$. So, in the basis $|0\rangle,\ \bar{\psi}|0\rangle$

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Also, ψ is an annihilation operator and $\bar{\psi}$ is a creation operator.

1.3.2. *Number operators.* These are the bosonic one $N_b = \bar{a}a$ (recall that for the harmonic oscillator $H = \frac{1}{2}(p^2 + x^2)$ we used the complex coordinate $a = \frac{1}{\sqrt{2}}(x + ip)$), and the fermionic one $N_f = \bar{\psi}\psi$. One has

$$[N_b, \bar{a}] = \bar{a}, \quad [N_b, a] = -a, \quad [N_f, \bar{\psi}] = \bar{\psi}, \quad [N_f, \psi] = -\psi,$$

and on $\bar{a}^n \bar{\psi}^\delta$, $N_b = n$ and $N_f = \delta$.

1.4. **Path integrals.** Operator $e^{-\beta H}$ corresponds to the interval I_β of length β, i.e., the interval $I = [0, 1]$, with the metric $g = \geq \widehat{g}$. Its trace, $Tr(e^{-\beta H})$ is the path integral for the circle S^1_β of length β:

$$Tr[e^{-\beta H}] = \int_M \Delta_a \langle a|e^{-\beta H}|a\rangle = \int_M \Delta_a \int_{x(\beta)=a=x(0)} Dx\ e^{-S[x,g]}$$

$$= \int_{x:\ S^1_\beta \to M} Dx\ e^{-S[x,g]} = \int_{x \in\ \Lambda(M)} Dx\ e^{-S[x,\beta\widehat{g}]}.$$

In the super version, M is a super vector space $M_0 \underset{\mathbb{R}}{\otimes} \mathbb{R}^{0|2}$, and we let $M_0 = \mathbb{R}$.

Now, following the above calculation with idealized position states $|x, \psi_1, \psi_2\rangle$ in M, the integration rules actually give

$$Tr[e^{-\beta H}] = \int_{x(\beta)=x(0),\ \psi_i(\beta)=-\psi_i(0)} Dx\ e^{-S[x,\psi_1,\psi_2,\beta\widehat{g}]}.$$

Problem: anti-periodic boundary conditions that are natural for the fermionic variables do not have SUSY – relation like $\delta x = \frac{1}{2}(\psi + \bar{\psi}) = \psi_1$ is a collision of periodic and anti-periodic boundary conditions.

Resolution: instead of the trace $Tr[e^{-\beta H}]$ consider the supertrace $Tr_F[e^{-\beta H}]$. The super trace is defined by $Tr_F(A) \stackrel{\text{def}}{=} Tr[(-1)^F A]$ where $(-1)^F$ is the parity encoded as ± 1. Then one reverts to the all-around periodic boundary conditions:

$$Tr_F[e^{-\beta H}] = \int_{x(\beta)=x(0),\ \psi_i(\beta)=\psi_i(0)} Dx\ e^{-S[x,\psi_1,\psi_2,\beta\widehat{g}]} = \int_{(x,\psi_1,\psi_2)\in\ \Lambda(M)} Dx\ e^{-S[x,\beta\widehat{g}]}.$$

2. **Case** $W = \frac{1}{2}x^2$

Then $W' = x$ and $W'' = 1$. So

$$L = L_E = \frac{1}{2}[\dot{x}^2 + x^2] + i\bar{\psi}\dot{\psi} + \frac{1}{2}(\bar{\psi}\psi - \psi\bar{\psi}) = \frac{1}{2}[\dot{x}^2 + x^2] + i\bar{\psi}\dot{\psi} + \bar{\psi}\psi.$$

2.0.1. *The variation.*

$$\delta L_E = \dot{x}\ \delta\dot{x} + x\ \delta x + i(\delta\bar{\psi}\ \dot{\psi} + \bar{\psi}\ \delta\dot{\psi}) + \delta\bar{\psi}\ \psi + \bar{\psi}\ \delta\psi.$$

2.1. **Canonical formalism.** We start with the conjugate momenta $x \mapsto p \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{x}} = \dot{x}$ and $\psi \mapsto \Pi \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{\psi}} = -i\bar{\psi}$. (However, $\bar{\psi} \mapsto \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0$?).

2.1.1. *Lemma.* (a) $\{p, x\} = 1$, $\{\bar{\psi}, \psi\} = -i$, while x, p commute with $\psi, \bar{\psi}$.

(b) At $\hbar = 1$, the non-trivial commutators are

$$[x, p] = i, \quad [\psi, \bar{\psi}] = 1 = [\bar{\psi}, \psi],$$

while x, p commute with $\psi, \bar{\psi}$, and $\{\psi, \psi\} = 0$ and $0 = \psi^2 = \bar{\psi}^2$.

Proof. We use the following rule: the non-zero Poisson brackets between coordinates come from the conjugate momenta, i.e., $1 = \{p, q\}$ means that $1 = \{\frac{\partial L}{\partial \dot{x}}, x\}$. In the same way $1 = \{\frac{\partial L}{\partial \dot{\psi}}, \psi\} = \{\Pi, \psi\} = -i\{\bar{\psi}, \psi\}$. This quantizes to $[\widehat{p}, \widehat{x}] = \frac{\hbar}{i}1$ and $[\widehat{\bar{\psi}}, \widehat{\psi}] = \frac{\hbar}{i}i$. Notice that $\{\bar{\psi}, \psi\} = \{\psi, \bar{\psi}\} = i$ gives $[\bar{\psi}, \psi] = [\psi, \bar{\psi}] = \hbar$.

Finally, for any odd quantity ψ, relation $\{\psi, \psi\} = 0$ gives $0 = [\widehat{\psi}, \widehat{\psi}] = 2\widehat{\psi}^2$.

2.1.2. *Hamiltonian.* We start with the Minkowski Lagrangian

$$L_M = \frac{1}{2}[\dot{x}^2 - x^2] + i\bar{\psi}\dot{\psi} - \frac{1}{2}(\bar{\psi}\psi - \psi\bar{\psi}),$$

the Hamiltonian is then

$$H = H_M = p\dot{x} + \Pi\dot{\psi} - L_M = \frac{1}{2}[\dot{x}^2 + x^2] + \frac{1}{2}(\bar{\psi}\psi - \psi\bar{\psi}).$$

2.2. **Symmetries of the action.**

2.2.1. *Lemma.* The Hamiltonian vector fields $\{-, Q_i\}$ given by the functions

$$Q_1 = \frac{1}{2}p(\psi + \bar{\psi}) \quad Q_2 = \frac{1}{2}p(\psi - \bar{\psi})$$

are infinitesimal symmetries of the action.

Proof. The vector fields $\delta_i = \{-, Q_i\}$ are given by

$$\delta_1 x = \frac{\varepsilon_1}{2}(\psi + \bar{\psi}), \quad \delta_1 \psi = \frac{\varepsilon_1}{2} i\dot{x}, \quad \delta_1 \bar{\psi} = \frac{\varepsilon_1}{2} i\dot{x},$$
$$\delta_2 x = \frac{\varepsilon_2}{2}(\psi - \bar{\psi}), \quad \delta_2 \psi = \frac{\varepsilon_2}{2}\dot{x}, \quad \delta_2 \bar{\psi} = \frac{\varepsilon_2}{2} - \dot{x}.$$

Now one plugs in these expressions into δL_E and obtains 0.

2.2.2. *Infinitesimal symmetries as operators on the Hilbert space.*

$$Q_1 = \frac{1}{2}p(\psi + \bar{\psi}), \quad Q_2 = \frac{1}{2}p(\psi - \bar{\psi}).$$

2.2.3. *Lemma.* (a) $i[Q_1, x] = \frac{1}{2}(\psi + \bar{\psi}), \quad i[Q_1, \psi] = \frac{1}{2} i\dot{x}, \quad i[Q_1, \bar{\psi}] = \frac{1}{2} i\dot{x}, .$
(b) $i[Q_2, x] = \frac{1}{2i}(\psi - \bar{\psi}), \quad i[Q_2, \psi] = \frac{1}{2}\dot{x}, \quad i[Q_2, \bar{\psi}] = \frac{1}{2} - \dot{x}.$

3. 1-dimensional QFT: σ-models

4. Geometry

4.1. **Anomaly.** Let us say that a classical symmetry has an anomaly if it does not survive the quantization. From the point of view of path integrals, a symmetry of the action still need not preserve the measure in the path integral (we have been vague on this measure), hence need not preserve the path integral.

This may be OK – a classical symmetry may be replaced by its own quantization. For the Polyakov action the diffeomorphism symmetry quantizes to the Virasoro symmetry, and the Weyl symmetry (i.e., the conformal change of metric), has an anomaly which cancels for $D = 26$.

4.2. **Spinors.** These are the sections of any spinor bundle S on Σ. Recall that we have (Σ, g, or), hence a complex structure $\mathcal{C}$ on Σ. A spinor bundle is any holomorphic line bundle which is a square root of the canonical bundle K. Now, spinors are the sections of a spinor bundle S on Σ.

Spinor bundles form a torsor for $H^1(\Sigma, \pm) \cong \mathbb{Z}_2^{2g}$.

Part VIII. Dictionaries

Appendix A. Dictionary

Notice that the IAS text has a useful dictionary.

A.0.1. *Action.* Classically, physical laws are expressed in the Lagrangian approach by requiring that the certain quantity $S[x]$ attached to a possible evolution x of the system is minimal. One calls $S[x]$ the action of x.

The action is usually the worldsheet integral (sometimes maybe a spacetime integral?), of the Lagrangian function (= action density). The physical setting is therefore encoded in the choice of the Lagrangian function.

Action is written in terms of elementary fields of the theory.[25]

A.0.2. *Euler-Lagrange equations.* These are the associated PDEs - critical points of the Lagrangian (=Action) as a function on some configuration space.

A.0.3. *(Non)Dynamical variables.* A field y will be said to be non-dynamical if it appears in the Lagrangian but its derivatives do not.

In quantization, dynamical variables give the *naive* Hilbert space $\mathcal{H}_{naive}$, and the non-dynamical ones then give the *constraints*, which carve out the physically meaningful part $\mathcal{H}_{ph} \subseteq \mathcal{H}_{naive}$.

Symmetry principle. In trying to describe a physical system one may have some latitude in choosing the action (Lagrange function) and even the fields in the action. It is expected that one should choose
cen the most general action compatible with all symmetries.

A.1. **Background.** A field that is kept fixed in the construction of a QFT.

For example, in a σ-model: the metric on the target.

A.2. **Bosons and fermions.** Consider an elementary particle and its Hilbert space of states $\mathcal{H}$. We say that this is a bosonic particle or Fermionic particle, if the Hilbert space of states of n particles of this kind is $S^n\mathcal{H}$ or if it is $\wedge^n\mathcal{H}$. We also say that in these cases the particle satisfies the Bose or Fermi statistics.

A.3. **Classical solutions.** The solutions of the EL-equations. i.e., the critical points.

[25] Sometimes the *Action* is called *Lagrangian*, and Lagrangian is called *Lagrangian density*.

A.4. **Correlation functions.** These are the fundamental objects of the theory. they are usually computed perturbatively, using the Feynman diagrams.

A correlation function is a diagonal matrix element for a composition of local operators

$$\langle v|\phi_1(x_1)\cdots\phi_n(x_n)|v\rangle = \langle\phi_1(x_1)\cdots\phi_n(x_n)v,\ v\rangle$$

for the vacuum $v¿$ more generally, it is a general matrix element between two vectors i.e., two states.

It is a distribution on the configuration space, the space of positions x_i of local operators ϕ_i.

The n-point function of a local field $\phi(x)$ is the correlation function

$$\langle\phi(x_1)\cdots\phi_n(x)v, v\rangle$$

A.4.1. *Expectation value.* Expectation value of $\phi(x)$ is a special correlation function

$$\langle\phi(x)v,\ v\rangle$$

i.e., the 1-point function!

A.4.2. They exist in the physically meaningful Lorentzian setting. Their Wick rotation is the Schwinger function.

A.5. **Critical dimensions.** For the ST critical dimension is 26, however it contains a tachyon (negative mass), so one passes to the super ST which has critical dimension 10.

A.6. **CFT to QFT.** In CFT the normal ordering trick suffices for quantizing products ("finite composite operator").

A.7. **Dimensional regularization.** One can study spaces of dimension $2+\varepsilon$. The infinities of the theory are then the poles at $\varepsilon = 0$ of a nicer theory.

A.8. **Dualities.** Duality is a coincidence of two CFT. A necessary condition for a coincidence is given by a dimension count (of total and compactified directions).

A.8.1. *Proofs.* However, proofs of duality seem to be ad hoc, i.e., calculation of both theories. This seems like the state of Langlands where large parts are proved by representation theory without understanding the mechanism of L-duality.

ST may only yield manifestations of duality rather then a framework for understanding it (that may be in M-theory).

A.9. **Free Theory.** No interactions. Oscillators uncoupled.

A.10. **Feynman graphs (diagrams).**

A.10.1. *Mathematics: expansion of integrals* $\int_V e^{-S}$. Feynman graphs index the terms in the expansion of integrals

$$\int_V e^{-S}$$

over a vector space V, and for a function which is quadratic plus higher terms.

The propagator is $(S_2)^{-1}$ the inverse of the quadratic part S_2 of S.

Vertices: higher terms of S (tensors on V^*).

Edges: contraction via the propagator.

Perturbation expansion of the integral: the sum of all amplitudes of Feynman graphs.

Amplitude of the Feynman graph: the answer after the contraction.

A.10.2. *Physics.* A Feynman diagram is a scenario of particle interaction.

Vertices: interactions of particles (governed by the the terms appearing in the Lagrangian).

Edges: Free motion of a particle in the spacetime.

Idea: Expectation values of quantities are sums over all histories, i.e., over all diagrams. Each process is to be counted with weight e^{-S} (Euclidean spacetime) or e^{iS} (Lorentzian), determined by the action S. For each history (i.e., graph), one should integrate over all positions of vertices. This expectation (correlation function) is a path integral.

A.11. **Field.** Any local data on a spacetime.

Classical example: section of a line bundle for a line bundle over the spacetime.

Quantum: Operator valued distribution on a space time, or linear map from smooth sections of a vector bundle over the spacetime to operators.

A.11.1. *Dynamical fields.* Those that appear in the action with the derivatives. In Polyakov action map is dynamical but the metric on the source is not.

A.11.2. *Non-dynamical fields.* They supply constraints on the remaining degrees of freedom hence reduce their phase space.

Say for the timelike component A_0 of the gauge field in YM one can say that the integration over the choices of A_0 gives a δ-distribution

$$\int e^{iA_0 x}\, dA_0 = \delta(x).$$

Constraints generate a group acting on the phase space and the quotient is called the reduced phase space.

A.12. **Field theory.** It is free if

QFT: the Hilbert space is the completed symmetric product (symmetric power) of the one particle space (as a representation of the P-group). Then the S-matrix vanishes and the correlation functions have a simple formula in terms of the inverse of the Laplace or the wave operator..

A.13. **Fock space.** This means S^*W for the Hilbert space W of one particle states!

Hopf algebra structure. A priori it is a Hopf algebra $\mathcal{O}(W^*) = U(W)$. Maybe these structures are meaningful ?

A.14. **σ-model.**

A.15. **Hamiltonian quantization as a quantization of classical solutions.** Consider an action S. Classical solutions are the critical points $dS = 0$ (the differential is called the variation in the infinite dimensional setting, denoted δS.) The Hamiltonian quantization will require a Hilbert space $\mathcal{H}$ and operators on $\mathcal{H}$.

Let us build $\mathcal{H}$ here by thinking of the classical solutions.

If there are several variables quantization can be done in stages. We start with the

A.15.1. *Dynamical variables x.* Solve $\delta_x S = 0$, i.e., fix the non-dynamical variable g and consider the variation of x only, find the critical x's (for a fixed g). Quantize the solutions - if the solution is a formula for x involving some parameters x_n then one quantizes x_n so that they become operators on a Hilbert space $\mathcal{H}$.

ST example. Here, $\delta_x S = 0$ means that x is a sum of a holomorphic and an anti-holomorphic function $x = x_L + x_R = sum\ x_n z^{-n-1} + sum\ \widetilde{x}_n \bar{z}^{-n-1}$.

A.15.2. *Non-dynamical variables g.* These integrate to an action of a group on the phase space. One has some choice in the approach to quantization.

(1) "At the end of the day impose the constraint on the Hilbert space. This will carve out the physical part."

So one starts with (a) quantize the system, then (b) impose the the quantized critical point equation $\boldsymbol{T} \stackrel{\text{def}}{=} \delta_g S = 0$ in the form of requiring that $\langle u, \boldsymbol{T}, v\rangle = 0$ for u, v in a subspace $\mathcal{F}^{physical}$ of physically meaningful states. This involves imposing $L_n = 0$ for half of the coefficients L_n of $\boldsymbol{T}$, hence a choice of a polarization of the Lie algebra generated by L_n's. (c) Among the possible choices in your construction, fix those that that make the inner product on $\mathcal{H}^{ph}$ semi-positive. (d) Now the null-directions make a vector subspace $\mathcal{H}^{null}$, the correct space is

$$\mathcal{H}^+ \stackrel{\text{def}}{=} \mathcal{H}^{ph}/\mathcal{H}^{null}.$$

(2) BRST: enlarge the Hilbert space $\mathcal{H}$ to a complex $(\mathcal{H}^*, d)$ by allowing also the odd directions. Then

$$H^*(\mathcal{H}^*, d) = \mathcal{H}^+.$$

(3) Solve the critical equation and then quantize. This involves a choice of directions in the phase space, transversal to the orbits of the group generated by our non-dynamical variable g. So it is not manifestly Lorentz equivariant.

ST example. The group generated by the non-dynamical variable g in the action is (?) $Diff(\Sigma)$.

A.15.3. *Instanton corrections.* Example of non-perturbative effects.

A.16. **Lagrangian.** See *Action.*

A.17. **Left and right movers.** For a bosonic free string one decomposes the motion of a string into a combination of elementary motions x_n. Say, $x_1(\sigma - \tau)$ means that if we want to be at the same displacement as time τ changes, we should change position σ in the same way. So an initial bump (initial position of a wave), gives a wave traveling to the right. In the same way, $x_1(\sigma + \tau)$ will mean that the bump travels to the left.

The same for x_n, $n \neq 0$, except that the frequency (or velocity) of the wave is multiplied by n. However, x_0 is different - it is the translation of the loop in the target vector space.

A.18. **Mass.** A simple Lagrangian density for a point particle is $|d\phi|^2 + m^2\phi^2$. So any term in a Lagrangian density which is quadratic, i.e.of the form $m^2\phi^2$ or $m^2 A_\mu A^\mu$ is called a mass term, and one says that the field has a mass m! So the mass of the field is the square root of the coefficient of the quadratic term in the Lagrangian density.

A.19. **Minimal surfaces.** ST seems to be a quantization of minimal surfaces (harmonic maps?) but mathematically it is not known what that would mean.

Can Physicists quantize harmonic equations, say the Hodge theory?

Hope: MS problems will become more rigid after quantization.

A.19.1. *Minimal surfaces are the classical solutions of the Polyakov action.* A minimal surface or say a geodesic, involves only a metric G on the target. So one requires both that the metric on Σ be conformal to the pull-back metric, and that a map be harmonic. In the case of a geodesic it corresponds to parameterizing geodesic curves by unit speed. For surfaces, again it may be that a harmonic map for (g, G) is not the right notion - that its image is a minimal surface but that the correct reparameterization (of "unit speed" or area preserving) is the one where g is conformal to x^*G.

The standard problem of harmonic maps is when one fixes g and G and looks for extremals. It is not the GN-action - S_{GN} is S_{po} with g is conformal to the pull-back!

Actually, since S_{po} depends only on the conformal class of g - the same as the complex structure on Σ - we can fix a complex structure rather then g.

A.20. **Monopoles.** These are the solutions of the classical equations that do not deform (?).

BPS condition requires that the monopol has some of the SUSY that the theory has. So it is viewed as avoidance of SUSY.

A.21. **Parafermions.** Particles with braiding $\mathbb{Z}_n$ rather then only $\mathbb{Z}_2$ as for fermions. This is where a generalization of vertex algebras appears.

A.22. **Physicists.** They reduce any problem to an old one (say the harmonic oscillator).

A.22.1. *Planck constant $\hbar$.* To simplify formulas one often uses a choice of units for which $\hbar = 1$. However, one will sometimes think of this as a variable. *Physically*, this corresponds to the idea that as we *change the scale* that we are interested in, our view of the size of $\hbar$ changes. *Formally*, it is a useful trick to have an *additional variable*.

A.23. **Positivity of inner products.** (i) The problem may be said to appear from the lack of the same in Minkowski metric.

(ii) It seems to correspond to the expansion of a physical; system from a stable critical point (i.e., a minimum, rather then a saddle). Therefore lack of positivity should be a natural feature of some situations.

A.24. **Nonlinear equations of motion.** One 1st treats the free theory (linear equations). This gives a Hilbert space and then one checks that the perturbed theory stays in this space.

A.25. **Quantum Mechanics.** It allows no interactions (creation ...). For that one need QFT. So the QM of ST is the case of a cylinder (strip).

A.26. **Quantization formalisms.**

A.26.1. *Lagrangian formalism: path integrals (functional integrals).* Correlation function of local fields $\phi_1(x_1), ..., \phi_n(x_n)$ is calculated as a path integral over the space of elementary field configuration.

$$\int \mathcal{D}\Phi \ \phi_1(x_1)\cdots\phi_n(x_n)\ e^{iS},$$

for the path integral measure $\mathcal{D}\Phi$ on the space of configurations of elementary fields.

A.26.2. *Hamiltonian: Operators on a Hilbert space.* It is traditionally local, by formulas, then one has to check whether it is globally meaningful.

A.26.3. *Feynman-Kac formula.* This is the coincidence of the two formalisms.

Holds in QM but in QFT only after both sides are renormalized.

A.27. **Quantization of the ST.**

A.27.1. *Hilbert space.* Consider the space of maps of a string d^* into M. Quantization uses "half" of info: just a map of $(d,0)$ into (M,pt)! So the classical "phase space" is $Map[d^*,M]$, it contains a Lagrangian $Map[(d,0),(M,pt)]$. It quantizes to the Hilbert space

$$\mathcal{H} = \mathcal{O}\left(Map[(d,0),(M,pt)]\right)$$

and the algebra of functions on $Map[d^*,M]$ quantizes to the differential operators on $Map[(d,0),(M,pt)]$.

There should be one Hilbert space for each marked point a. i.e., each external particle. It should be

$$\mathcal{H}_a = \mathcal{O}\left(Map[(\widehat{a},a),(M,pt)]\right)$$

but this rather suggests having one Hilbert bundle $\mathcal{H}_a$ over the spacetime, with fibers

$$\mathcal{H}_a(m) = \mathcal{O}\left(Map[(\widehat{a},a),(M,m)]\right).$$

However, then one should have some differential equation such that the global phenomena are the same as local (initial value problem), at least in the absence of topology on M!

The biggest geometric object around is the space $Jet_\Sigma(M)$ of jets of maps from Σ to M (i.e. of sections of $M_\Sigma \stackrel{\text{def}}{=} M\times\Sigma\to\Sigma$. It is a bundle over Σ and the fiber at $a\in\Sigma$ is $Jet_{\Sigma,a}(M)\stackrel{\text{def}}{=} Map(\widehat{a},M)$. Moreover, the fiber maps to M by evaluating the map at the center of the disc. Of course, we could also vary Σ.

To marked surface Σ, a_i one can associate the commutative algebra

$$\mathcal{O}(Jet_\Sigma(M)|\{a_i,\ i\in I\}) = \mathcal{O}(\prod_{i\in I},\ Jet_\Sigma(M)|\{a_i\}) = \mathcal{O}(\prod_{i\in I},\ Map(\widehat{a}_i,M)\}),$$

and if we also have a map $\{a_i,\ i\in I\}\stackrel{x_\infty}{\longrightarrow} M$, we get the quotient algebra of functions on the space

$$Map[(\sqcup\,\widehat{a}_i,\{a_i,\ i\in I\}),(M,\{x_\infty(a_i),\ i\in I\}] = \prod_{i\in I} Map[(\widehat{a}_i,a_i),(M,x_\infty(a_i))]$$

of extensions of x_∞ to a formal neighborhood in Σ.

A.27.2. *Negative norm and zero norm.* Example: timelike oscillations.

These are avoided since physicists think of norm as a probability. The ghost fields cause cancellation of this unphysical part.

So a larger space with a pathology reduces to a physically meaningful one.

A.27.3. *Spectrum of String Theory (= "physical excitations").* This is the Hilbert space of the constrained quantum system.

A.28. **Relativity.** It is kept on the target space but not on the world sheet.

A.29. **Spontaneous symmetry breaking.** It arises when one studies a system with a symmetry G by perturbation expansion around a state that does not have that symmetry.

Example: Double well potential. Let the potential be an even function of the position x, with the minimum around ± 1 and a local maximum at 0. Then the theory has a ± 1 symmetry. Physicists do expansions around local minima (stable) and the expansion looses the global symmetry.

Deeper example. Suppose that the "potential" is a vector v in a representation of G. If we do expansion around a particular v, the remaining symmetry is G_v. If v is say an extremal vector one keeps a Cartan symmetry, for a generic vector all symmetry may go.

Spontaneous symmetry breaking may add mass to a massless particle: expansion of the Lagrangian around v may add a quadratic term $|v|^2\phi$.

A.30. **State field correspondence.** A mystery?

A.31. **Renormalization.** Gives a family of QFT from one Lagrangian.

A.32. **Spacetime.** Examples when it is not split into space times time: exist, as compact Riemannian manifolds ("Euclidean signature"). Such theories can be done in the Lagrangian framework. However in the Hamiltonian framework (operators with a Hamiltonian) one needs to separate time.

A.32.1. *Time.* In the string formalism, time does not live on a worldsheet Σ, it is a function on the target M and it becomes a function on Σ once one fixes $x : \Sigma \to M$!

A.33. **Vertex operators.** Expectation value or amplitude can be associated to a a Feynman (string) diagram, symbolizing several incoming particles interacting and scattering, so it will depend on the states of the external particles. A state of a particle is a vector, and by the state–field correspondence in CFT it is an operator.

The state–field correspondence as a bijection may not hold for more general QFT, or at least is deemphasized.

Let the action be

$$S = \int \partial_z \phi \cdot \partial_{\overline{z}} \phi$$

. The basic vertex operator is the tachyon

$$V_\psi(z) = : e^{ip\phi(z)} :$$

with

$$\phi = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n} + \overline{\alpha}_n \overline{z}^{-n}.$$

The conformal weight of a field is

$$h = \frac{p}{2}.$$

Here $p^2 = 2 = -m$.

For instance consider two incoming tachyons which interact and produce two outgoing tachyons, the change may be a mixing of momenta. Then one has a zero-loop diagram which is a cross on the Feynman level and on the string level it is a sphere with four semi-infinite tubes sticking out. Then one uses a conformal transformation to interpret this as a sphere with 4 punctures. The state of an incoming tachyon can be thought of as a vector and the corresponding vertex operator as a disturbance along the tube.

A.34. **Virasoro.** Moving a vector by Virasoro generators corresponds to a conformal transformation.

A.35. **Various.**

A.35.1. *Parameter ls.* It appears in the Polyakov action to make it dimensionless so that the path integral is defined.

A.35.2. *Level.* This is the Central charge C and it will equal the Dimension of the target D. It is also called the conformal anomaly since in the path integral formulation of QM it is the failure of QM to be conformally invariant.

Appendix B. **Constants**

D, d, T, c Units.

Part IX. **Mathematics**

Appendix C. **Coordinates**

We describe some standard coordinate conventions.

(1) The covariant objects have lower indices:
- Bases vectors e_i in a vector space V,
- Coordinates α_i of linear functionals $\alpha \in V^*$,
- Partial derivatives ∂_μ,

and the contravariant objects have upper indices:
- Dual bases vectors e^i of V^*
- Coordinates v^i of a vector $v \in V$,

- Coordinate functions x^μ.

(2) The upper and lower indices contract
- $v = v^i e_i$,
- The coordinates of inverse matrices A and A^{-1} usually have the opposite position of indices $A = (A^{ij})$ and $A^{-1} = (A_{ij})$.

(3) Lowering and lifting of indices is done using a metric. A metric g on a vector space V gives an identification $io_g : V \xrightarrow{\cong} V^*$. Now,
- for a vector $v \in V$, writing v with the lower indices v_i means the coordinates of the linear functional $\iota_g v \in V^*$.
- One example is the calculation of a metric in a fake diagonal form

$$g(u, v) = u^i v^j g_{ij} = u^i v_i.$$

(4) Applying a quadratic form g to a basis e_i gives lower indices $g_{ij} = g(e_i, e_j)$.

In the world of manifolds, the basic covariant linear algebra object is the tangent bundle TM while T^*M is contravariant. So the standard position of indices is

- partial derivatives are denoted ∂_μ,
- while the coordinate functions are x^μ,
- a metric on M (i.e., on TM) is given by a matrix (g_{ij}), and the inverse metric $g^* = g^{-1}$ on T^*M is given by the inverse matrix (g^{ij}).

C.1. **Coordinates of vectors.**

C.1.1. *Coordinates of vectors.* A basis e_i of a vector space V gives coordinates on V: $v = v^i e_i$, and the dual basis e^i of V^* gives coordinates in V^*: $\alpha = \alpha_i e^i$.

For an (ordered) bases $\mathcal{A}$ of V we denote by $v^{\mathcal{A}}$ the coordinate column vector of a vector $v \in V$ in the basis $\mathcal{A}$. For another ordered bases $\mathcal{B}$ of V we denote

- by $({}_{\mathcal{B}}\leftarrow_{\mathcal{A}})$ the change of bases matrix,
- by $({}^{\mathcal{B}}\leftarrow^{\mathcal{A}})$ the coordinate change matrix,

in the sense of

$$b_i = ({}_{\mathcal{B}}\leftarrow_{\mathcal{A}})_{ij} \cdot a_j \quad \text{and} \quad v^{\mathcal{B}} = ({}^{\mathcal{B}}\leftarrow^{\mathcal{A}}) \cdot v^{\mathcal{A}}.$$

For instance, $({}_{\mathcal{B}}\leftarrow_{\mathcal{A}})_{ij} = \langle b_i, a^j \rangle$.

Lemma. (a) If $V = \Bbbk^n$ then an ordered bases $\mathcal{A}$ defines a matrix which we denote $\underline{\mathcal{A}}$ (the columns are the bases vectors). In this case

- (i) $v = \underline{\mathcal{A}} \cdot v^{\mathcal{A}}$,
- (ii) the change of coordinates matrix is now $({}^{\mathcal{B}}\leftarrow^{\mathcal{A}}) = \underline{\mathcal{B}}^{-1} \cdot \underline{\mathcal{A}}$,
- (iii) the dual basis $\mathcal{A}^*$ of $\Bbbk^n$ is given by the matrix $\underline{\mathcal{A}^*} = \underline{\mathcal{A}}^{-tr}$.

(b) The columns of the matrix $(\mathcal{B} \leftarrow \mathcal{A})$ are the $\mathcal{B}$-coordinates of vectors in $\mathcal{A}$.

(c) The change of bases matrix $({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}})$ is the inverse of the change of coordinate matrix

$$({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}}) = ({}^{\mathcal{B}}\!\leftarrow^{\mathcal{A}})^{-1} = ({}^{\mathcal{A}}\!\leftarrow^{\mathcal{B}}).$$

(d) The change for dual basis is the transpose inverse of the change for original basis:

$$({}_{\mathcal{B}^*}\!\leftarrow_{\mathcal{A}^*}) = ({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}})^{-tr}.$$

Proof. (a) The i^{th} vector a_i in $\mathcal{A}$ is the i^{th} column of $\underline{\mathcal{A}}$, i.e., $a_i = \underline{\mathcal{A}} \cdot e_i$.

- (i), $\underline{\mathcal{A}} \cdot v^{\mathcal{A}} = a_i v^i = v$.
- (ii) Therefore, $\underline{\mathcal{B}} \cdot v^{\mathcal{B}} = v = \underline{\mathcal{A}} \cdot v^{\mathcal{A}}$, and then $v^{\mathcal{B}} = \underline{\mathcal{B}}^{-1} \underline{\mathcal{A}} \cdot v^{\mathcal{A}}$.
- (iii) $\delta_{ij} = \langle a_i^*, a_j \rangle = (a_i^*)^{tr} \cdot a_j = [(\underline{\mathcal{A}^*})^{tr} \underline{\mathcal{A}}]_{ij}$.

(b) $(\mathcal{B} \leftarrow \mathcal{A})_i = (\mathcal{B} \leftarrow \mathcal{A})\, e_i = (\mathcal{B} \leftarrow \mathcal{A})\, a_i{}^{\mathcal{A}} = a_i{}^{\mathcal{B}}$.

(c) The j^{th} column of $({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}})$ is $({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}}) \cdot e_j = (\phi_{*,j})$ and this is the coordinate vector $b_j{}^{\mathcal{A}} = ({}^{\mathcal{A}}\!\leftarrow^{\mathcal{B}}) \cdot b_j{}^{\mathcal{B}} = (\mathcal{A} \leftarrow \mathcal{B}) \cdot e_j$.

(d) $b_j = ({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}})_{jk} \cdot a_k$, hence $({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}})_{jk} = \langle b_j, a^k \rangle$. Therefore, $({}_{\mathcal{B}^*}\!\leftarrow_{\mathcal{A}^*})_{pq} = \langle b^p, a_q \rangle$, and

$$[({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}})^{tr}]_{jk} \cdot ({}_{\mathcal{B}^*}\!\leftarrow_{\mathcal{A}^*})^{kl} = ({}_{\mathcal{B}}\!\leftarrow_{\mathcal{A}})_{kj} \cdot ({}_{\mathcal{B}^*}\!\leftarrow_{\mathcal{A}^*})^{kl} = \langle b_k, a^j \rangle \cdot \langle b^k, a_l \rangle = \big\langle \langle b^k, a_l \rangle \cdot b_k, a^k \big\rangle = \langle a_l, a^k \rangle = \delta_l{}^k.$$

C.1.2. *Bilinear forms.* A bilinear form g on V is encoded by a matrix

$$g_{ij} \stackrel{\text{def}}{=} g(e_i, e_j).$$

We notice the following constructions

- *Map ι_g.* Form g gives a map $\iota = \iota_g \stackrel{\text{def}}{=} g(v, -) : V \to V^*$. We will use the notation v_i for the coordinates of $\iota_g v$ in the dual bases e^i.
- *Inverse bilinear form.* If ι is non-degenerate then ι_g is an identification of vector spaces, so it can be used to carry over g to V^*. The resulting form on V^* is called the inverse bilinear form and denoted $g^{-1} = g^*$. For $\alpha, \beta \in V^*$

$$g^*(\alpha, \beta) \stackrel{\text{def}}{=} g(\iota^{-1}\alpha, \iota^{-1}\beta) = \langle \alpha, \iota^{-1}\beta \rangle.$$

- *Bilinear forms on $V \otimes U$ and* $\mathrm{Hom}(V, U)$. If one has bilinear forms g and G on V and U, we get $g \otimes G$ on $V \otimes U$. If g is non-degenerate and $\dim V < \infty$, we have a form $g^* \otimes G$ on $V^* \otimes U \cong \mathrm{Hom}(V, U)$.

Lemma. (a) The map $\iota = \iota_g$ is given in coordinates by

$$v_i \stackrel{\text{def}}{=} (\iota v)_i = v^j g_{ji}, \qquad v \in V.$$

In particular, $\iota_g(e_i) = g_{ij}\, e^j$.

(b) The matrix (g^{ij}) of g^* (i.e., $g^*(e^i, e^j) = g^{ij}$), is the inverse of the matrix (g_{ij}) of g.

(c) If we encode g using two basis $\mathcal{A}$ and $\mathcal{B}$ then

$$g_{\mathcal{B}} = (_{\mathcal{B}}\!\leftarrow\!_{\mathcal{A}}) \cdot g_{\mathcal{A}} \cdot (_{\mathcal{B}}\!\leftarrow\!_{\mathcal{A}})^{tr} \quad \text{and} \quad (g^{-1})_{\mathcal{B}^*} = (_{\mathcal{B}}\!\leftarrow\!_{\mathcal{A}})^{-tr} \cdot (g^{-1})_{\mathcal{A}^*} \cdot (_{\mathcal{B}}\!\leftarrow\!_{\mathcal{A}})^{-1}.$$

(d) The matrix of $g\otimes G$ on $V\otimes U$ is the outer tensor product of the matrices for g and G, i.e, if e_i and f_μ are bases of V and U, then $(g\otimes G)_{i\mu,j\nu} = g_{ij}\cdot G_{\mu\nu}$.

(e) If g is invertible, for linear operators $V \xrightarrow{A,B} U$,

$$(g^*\otimes G)(A,B) = G(Ae_i, Be_j)\cdot g^{ij}.$$

In particular $||A||^2_{G,g} \stackrel{\text{def}}{=} (g^*\otimes G)(A,A) = Tr[g^{-1}\cdot A^*G]$ and $||A||^2_{G,A^*G} = \text{rank}(A)$.

Proof. (a) We have $\iota_g v = v_i e^i$, hence

$$v_i = \langle \iota v | e_i\rangle \stackrel{\text{def}}{=} g(v,e_i) = g(v^j e_j, e_i) = v^j g_{ji}.$$

In particular, $\iota e_i = (\iota e_i)_j\ e^j = (e_i)^k g_{kj}\ e^j = \delta_{ik} g_{kj}\ e^j = g_{ij}\ e^j$.

(b) $g_{ki}\cdot g^*(e^i,e^j) = g^*(g_{ki}e^i, e^j) = g^*(\iota e_i, e^j) = \langle e_i | e^j\rangle = \delta_{ij}$.

(c) Let $\phi = (_{\mathcal{B}}\!\leftarrow\!_{\mathcal{B}})$. The change from $\mathcal{A}\otimes\mathcal{A}$ to $\mathcal{B}\otimes\mathcal{B}$ is by $\phi\otimes\phi$, i.e., $b_i\otimes b_j = \phi_{i\mu}a_\mu\otimes\phi_{j\nu}a_\nu = \phi_{i\mu}\phi_{j\nu}\cdot a_\mu\otimes a_\nu$. So, for a quadratic form g on V,

$$(g_{\mathcal{B}})_{ij} = g(b_i,b_j) = \phi_{i\mu}\phi_{j\nu}\cdot g(a_\mu,a_\nu) = (\phi\cdot g_{\mathcal{A}}\cdot\phi^{tr})_{ij}.$$

Since $(g^*)^{\mathcal{A}^*} = (g_{\mathcal{A}})^{-1}$ and $(_{\mathcal{B}^*}\!\leftarrow\!_{\mathcal{A}^*}) = (_{\mathcal{B}}\!\leftarrow\!_{\mathcal{A}})^{-tr}$, the claim for g^* is the inverse of the claim for g.

(d) $(g\otimes G)_{i\mu,j\nu} = (g\otimes G)(e_if_\mu, e_jf_\nu) \stackrel{\text{def}}{=} g(e_i,e_j)\cdot G(f_\mu,f_\nu) = g_{ij}\cdot G_{\mu\nu}$.

(e) It suffices to check on a basis $(e^i\otimes f_\mu)(v) \stackrel{\text{def}}{=} \langle e^i|v\rangle\cdot f_\mu$, of $\text{Hom}(V,U)$, but

$$(g^*\otimes G)(e^i\otimes f_\mu, e^j\otimes f_\nu) = g^{ij}\cdot G_{\mu\nu}$$

and

$$g^{pq}\cdot G([e^i\otimes f_\mu]e_\psi, [e^j\otimes f_\nu]e_q) = g^{pq}\cdot\delta_{ip}\delta_{jq}G(f_\mu,f_\nu) = g^{pq}G_{\mu\nu}.$$

In particular,

$$||A||^2_{G,g} = (g^*\otimes G)(A,A) = G(Ae_i,Ae_j)\cdot g^{ij} = (A^*G)_{ij}\cdot g^{ji} = (A^*G\cdot g)_i^{\ i} = Tr[A^*G\cdot g] \quad \text{and} \quad ||A||^2_{G,A^*G}$$

C.1.3. *Derivatives.*

Lemma. (a) The *derivatives of bilinear and quadratic forms* are given in coordinates by

$$(\partial_{v,\mu}g)(u,v) = (\iota_u)_\mu = u_\mu \quad \text{and} \quad (\partial_{v,\mu})g(v,v) = 2v_\mu.$$

(b) The *differential of the determinant* at an invertible matrix $A = (A^{ij})$ with the inverse $A^{-1} = (A_{ij})$ is

$$d\det(A) = \det(A)\cdot A_{ji}\cdot dA^{ij}, \quad i.e., \quad \frac{\partial\det(A)}{\partial A^{ij}} = \det(A)\cdot A_{ji}.$$

Proof. (a) Since $g(u,-)$ is linear, $\partial_v g(u,v) = g(u,-) = \iota_u$ and $\partial_v g(v,v) = g(v,-) + g(-,v) = 2g(v,-) = 2\iota_v$. In coordinates, $(\partial_{v,\mu} g)(u,v) = (\iota_u)_\mu = u_\mu$. and $(\partial_{v,\mu})g(v,v) = 2v_\mu$.

Since $g(u,-)$ is linear, $\partial_v g(u,v) = g(u,-) = \iota_u$, or in coordinates $\partial_{v\mu} g(u,v) = (\iota_u)_\mu = u_\mu$. Consequently, $\partial_v g(v,v) = 2g(v,-) = 2\iota_v$, and $\partial_{v\mu} g(v,v) = 2v_\mu$.

(b) The partial derivative $\frac{\partial}{\partial A^{ij}} \det(A)$ is the cofactor α_{ij}. Now recall that $\sum_i A^{ij}\alpha_{ik} = \delta_{jk} \det(A)$, i.e., $A^{tr} \cdot \alpha = \det(A) \cdot I$. So, $\frac{\partial}{\partial A^{ij}} \det(A) = \alpha_{ij} = [\det(A) \cdot (A^{-1})^{tr}]_{ij} \det(A) \cdot A_{ji}$.

C.2. **Coordinates on manifolds.** Let M be a smooth manifold. A chart on an open part U of M is an isomorphism $\phi : U \xrightarrow{\cong} V$ with some V open in $\mathbb{R}^n$. We say that the component functions $X^1, ..., X^n$ of $\phi = (X^1, ..., X^n)$ are coordinates X^μ on U. Let $\psi = \phi^{-1} : V \xrightarrow{\cong} U$. We will use coordinates X^μ on U to identify U with an open part V of $\mathbb{R}^n$.

Coordinates give frames dX^μ and $\frac{\partial}{\partial X^\mu}$ of T^*M and TM over U. A tangent vector $v \in T_m M$ and a cotangent vector $\alpha \in T_m^* M$ now have coordinates

$$v = \sum_\mu v^\mu \cdot \frac{\partial}{\partial X^\mu} = v^\mu \frac{\partial}{\partial X^\mu}, \qquad \alpha = \sum_\mu \alpha_\mu \cdot dX^\mu = \alpha_\mu dX^\mu.$$

C.2.1. *Metrics.* By a metric g on a manifold M we will mean a collection of symmetric and non-degenerate bilinear forms on tangent spaces of M. So we do not assume that a metric has to be positive definite.

A metric g on M is encoded on U in a symmetric matrix of functions

$$g_{\mu\nu} \stackrel{\text{def}}{=} g(\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\nu}).$$

A metric g on M i.e., on the vector bundle TM, identifies TM and T^*M by $\iota_g v = g(v,-)$, which is described by

$$v_\mu \stackrel{\text{def}}{=} (\iota v)_\mu = g_{\mu\nu} v^\nu.$$

This notation really means that we identify TM and T^*M via ι.

C.2.2. *Inverse metric $g^{-1} = g^* on T^*M$.* Via ι_g we define a metric g^* on T^*M. The inverse matrix $(g^{\mu\nu}) \stackrel{\text{def}}{=} (g_{\mu\nu})^{-1}$ encodes g^* by $g^*(dX^\mu, dX^\nu) = g^{\mu\nu}$.

C.2.3. *Upper and lower indices.* Starting from $\frac{\partial}{\partial x^\mu}$ and dx^μ, lower indices correspond to covariant tensors (tensor powers of TM) and lower indices to contravariant tensors (tensor powers of T^*M). Upper and lower indices contract in calculations by pairing dual vector bundles T^*M and TM. However, the identification of TM and T^*M by the metric leads to the notational conventions of *lowering and lifting the indices.*

C.2.4. *Pull-back of a metric.* The pull-back of a metric G on M via a map $x: \Sigma \to M$ is a collection of bilinear forms $(x^*G)_a$ on tangent spaces $T_a\Sigma,\ a \in \Sigma$,

$$(x^*G)_a(u,v) \stackrel{\text{def}}{=} G(d_a x\ u, d_a x\ v), \quad u, v \in T_a\Sigma,$$

i.e., one pushes forward the tangent vectors via the differential of x and then evaluates G. If one is lucky, x^*G is again a metric. (A necessary condition: x is an immersion, i.e., the differentials $d_a x$ are injective. If G is positive definite this is also sufficient.)

If one has coordinates τ^i on Σ, the differential is encoded into the Jacobian matrix

$$\frac{\partial x^\mu}{\partial \tau_j} = dx^\mu(\partial_{\tau_j}) = \langle dx(\partial_{\tau_j}), dX^\mu \rangle.$$

The matrix of the pull-back metric $g = x^*G$ is

$$g_{ij} = (x^*G)(\partial_{\tau_i}, \partial_{\tau_j}) = G(dx\ \partial_{\tau_i}, dx\ \partial_{\tau_j}) = G(\frac{dx^\mu}{\partial \tau_i}\partial_{X^\mu}, \frac{dx^\nu}{\partial \tau_j}\partial_{X^\nu})$$

$$= G_{\mu\nu} \cdot \frac{dx^\mu}{\partial \tau_i}\partial_{X^\mu} \cdot \frac{dx^\nu}{\partial \tau_j}\partial_{X^\nu}.$$

C.2.5. *Paths.* A map $x: \Sigma \to M$ is described in U via its component functions $x^\mu = X^\mu \circ x$. We think of these as functions (i.e., *fields*) on Σ.

C.3. **Minkowski spacetime.** For $D = d + 1$, the D-dimensional Minkowski spacetime is

$$M = \ time \times space = \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^D,$$

with the Minkowski metric η corresponding to the quadratic form η on M given by

$$\eta \stackrel{\text{def}}{=} -x_0^2 + \sum_{i>0} x_i^2,$$

so it is not positive definite. More precisely, $\eta = -c^2 x_0^2 + \sum_{i>0} x_i^2$ for the light speed c, however we choose units so that $c = 1$. The corresponding matrix is

$$(\eta_{\mu\nu}) = (\eta^{\mu\nu}) = diag(-1, 1, ..., 1).$$

We keep the notation $u \cdot v = \eta(u,v)$ and $u^2 = u \cdot u$ as in the Euclidean case. However, the size of a tangent vector $v \in T_m M = M$ (velocity vector), will be now $\sqrt{-v^2}$ since we will be interested in vectors v with $v^2 < 0$. Physically this means that the velocity in space $\frac{\sqrt{\sum_{i>0} v_i^2}}{v_0}$ is $< 1 = c$.

This will introduce a minus sign in a number of places, say

- the kinetic energy is now $T = -\frac{m}{2}\dot{x}^2 = -\frac{\langle p|\dot{x}\rangle}{2}$,
- the canonical momentum is $p \stackrel{\text{def}}{=} -\frac{\partial L}{\partial \dot{x}} = -L_{\dot{x}}$,
- and the relation between the Lagrangian and Hamiltonian is $L + H = -\langle p|\dot{x}\rangle$.

Appendix D. **Metrics, connections, curvature**

We list basic differential constructions related to a metric on a manifold.

Let g be a metric on a manifold M, i.e., a non-degenerate bilinear form on TM. We calculate in local coordinates x^i.

D.1. **Levi-Civita connection.**

D.1.1. *Lemma.* (a) Metric g gives a canonical connection ∇ (called the *covariant differentiation* or *Levi-Civita connection*), on the tangent vector bundle, i.e., a way to define the derivative $\nabla_v u$ of a vector field u in the direction of a vector field v. It is characterized by
(1) The torsion $\mathcal{T}(u,v) \stackrel{\text{def}}{=} \nabla_u v - \nabla_v u - [u,v]$ vanishes, and
(2) g is flat for the corresponding connection on $T^*M \otimes T^*M$.
So, the requirements are

$$\nabla_u v - \nabla_v u = [u,v] \qquad \text{and} \qquad z\, g(u,v) = g(\nabla_z u, v) = g(u, \nabla_z v).$$

(b) More, explicitly, the connection is characterized by

$$2{\cdot}g(\nabla_x y, z) = x\, g(y,z) + y\, g(z,x) - z\, g(x,y) - g(x,[y,z]) + g(y,[x,z]) + g(z,[x,y])$$

D.2. **Christoffel symbols** $\Gamma_{ij}{}^k$**.**

The connection ∇ is encoded in Christoffel symbols

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}{}^k \partial_k.$$

We will calculate these in general and in terms of an orthonormal frame for TX.

D.2.1. *Lemma.* $2{\cdot}\Gamma_{ij}{}^p = (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}){\cdot}\, g^{lp}$.

Proof. By the part (b) of the lemma D.1.1, Christoffel symbols satisfy

$$2{\cdot}\Gamma_{ij}{}^k g_{kl} = 2{\cdot}g(\Gamma_{ij}{}^k \partial_k, \partial_l) = 2{\cdot}g(\nabla_{\partial_i}\partial_j, \partial_l)$$

$$= \partial_i\, g(\partial_j, \partial_l) + \partial_j\, g(\partial_l, \partial_i) - \partial_l\, g(\partial_i, \partial_j) - g(\partial_i, [\partial_j, \partial_l]) + g(\partial_j, [\partial_i, \partial_l]) + g(\partial_l, [\partial_i, \partial_j])$$

$$= \partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}.$$

It remains to multiply with the inverse matrix (g^{lp}).

D.2.2. *Lemma.* (a) If g is Riemannian then locally there is a frame v_i of TX which is orthonormal for g.

(b) Relative to an orthonormal frame v_i, Christoffel symbols $\nabla_{v_i} v_j = \Gamma_{ij}{}^k v_k$ are given by

$$2\Gamma_{ij}{}^l = - g(v_i, [v_j, v_l]) + g(v_j, [v_i, v_l]) + g(v_l, [v_i, v_j]).$$

Proof. (a) Apply Gramm-Schmid formulas to an arbitrary frame. For (b), $2\Gamma_{ij}{}^{l} = 2g(\Gamma_{ij}{}^{k}v_k, v_l) = 2g(\nabla_{v_i}v_j, v_l)$ equals

$= v_i\ g(v_j, v_l) + v_j\ g(v_l, v_i) - v_l\ g(v_i, v_j) - g(v_i, [v_j, v_l]) + g(v_j, [v_i, v_l]) + g(v_l, [v_i, v_j])$.

D.3. **Curvature ("Riemann tensor").** The curvature R of the connection ∇ is a 2-form with values in $\mathrm{End}(TM)$,

$$R(u, v) \stackrel{\mathrm{def}}{=} \nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{[u,v]},$$

for vector fields u, v on M. It is encoded by the Riemann tensor

$$R(\partial_i, \partial_j)\partial_k = (\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i} - \nabla_{[\partial_i,\partial_j]})\ \partial_k = R_{ijk}{}^{l}\ \partial_l.$$

D.3.1. *The symmetries of the Riemann tensor.*

Lemma. (a) $R(v, u) = -R(u, v)$.

(b) $R(u, v)$ is a symmetric operator for g

$$g(R(u, v)x, y) = g(x, R(u, v)y).$$

(c) If one interprets $R(-,-)$ a section of $T^*M\otimes T^*M\otimes\mathrm{End}(TM) = T(^*M)^{\otimes 2}\otimes T^*M\otimes T^*M\otimes TM$ as a section of $T(^*M)^{\otimes 2}\otimes T(^*M)^{\otimes 2}$ using g, the R is symmetric:

$$g(R(u, v)x, y) = g(R(x, y)u, v).$$

(d) A kind of a Jacobi identity

$$R(u, v)z + R(v, z)u + R(z, u)v = 0.$$

D.4. **Ricci curvature ("Ricci tensor").** Since the curvature R of the connection ∇ is a 2-form with values in $\mathrm{End}(TM)$, it can be interpreted as an operator $\mathcal{R} : TM\otimes TM \to \mathrm{End}(TM)$. Ricci curvature is the trace of this operator, hence a section of $T^*M\otimes T^*M$. Precisely, for vector fields u, v let $\mathcal{R}(u, v)z \stackrel{\mathrm{def}}{=} R(v, z)u$, i.e., $\mathcal{R}(u, v) = R(v, -)\otimes u$, then the Ricci curvature is

$$r(u, v) \stackrel{\mathrm{def}}{=} Tr[\mathcal{R}(u, v)] = \langle\mathcal{R}(u, v)\partial_k|dx^k\rangle = \langle R(v, \partial_k)u|dx^k\rangle.$$

It is encoded by the Ricci tensor

$$r_{ij} \stackrel{\mathrm{def}}{=} r(\partial_i, \partial_j) = \langle R(\partial_j, \partial_k)\partial_i|dx^k\rangle = \langle R_{jki}{}^{l}\partial_l|dx^k\rangle = R_{jki}{}^{k}.$$

D.4.1. *Lemma.* Ricci tensor is symmetric, $r(v, u) = r(u, v)$, i.e., a metric-like object (Since the Riemann tensor, interpreted as an operator is symmetric.)

D.5. **Laplace operator.** The Laplace operator for the metric g on Σ is

$$\nabla_g^2 \stackrel{\mathrm{def}}{=} g^{ij}\partial_{\tau_i}\partial_{\tau_j} = g^{ij}\partial_i\partial_j.$$

APPENDIX E. **Surfaces: relation between metrics and complex structures**

We will see that on an oriented surface a complex structure J is the same as a conformal class of metrics g. We say that g is *conformally flat* for J (or for any local holomorphic coordinate z).

What this means for us is that it is convenient to do computations for a given metric g using the associated complex structure J. So, metric questions become questions in holomorphic geometry.

E.1. **Metrics and complex structures on surfaces.**

E.1.1. *Lemma.* (a) A complex structure on a surface Σ is the same as $J: T\Sigma \to T\Sigma$ with $J^2 = -1$.

(b) A metric g on an orientable surface Σ defines a pair of opposite complex structures on Σ. A choice of an orientation *or* of Σ then picks one of these.

(c) With a fixed orientation, two metrics give the same complex structure iff they are conformally equivalent.

Proof. (a) A complex structure clearly gives such J. Conversely, ...

(b) If one requires that the operator J_a on $T_a\Sigma$ is orthogonal then there are two solutions for $J_a^2 = -1$, and they correspond to the two orientations of Σ at a.

(c) J_a determines g_a up to a scalar – for a fixed vector e, asking that the length is 1 fixes a multiple cg_a of g_a, then $J_a e, e$ is an orthonormal basis for cg_a, so they determine cg_a.

E.2. **Passage to complex coordinates.** We will consider a surface Σ with global smooth coordinates τ^1, τ^2 (so one can think of Σ as an open part of $\mathbb{R}^2$). Two real coordinates can be viewed as one complex coordinate $z = \tau^1 + i\tau^2$. So they define a complex structure on Σ such that z is a holomorphic coordinate and $\bar{z} = \tau^1 - i\tau^2$ is an anti-holomorphic function.

Here we just list the formulas for a metric g and the inverse metric g^{-1}, that relate the real coordinates τ^i to $z, \bar{z}$.

E.2.1. *Notational conventions.* We will denote a holomorphic function by $\phi(z)$, an anti-holomorphic function by $\psi(\bar{z})$, and a function which is neither by $f(z, \bar{z})$ (though $\bar{z}$ contains no new information). We will use

- the frame $\partial_i = \partial_{\tau^i}$ of $T\Sigma$ given by the real coordinates, and
- the frame of the complexified tangent bundle $(T\Sigma)_{\mathbb{C}}$ associated to the (anti)holomorphic coordinates $z, \bar{z}$

$$\partial \stackrel{\text{def}}{=} \partial_z \stackrel{\text{def}}{=} \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \bar{\partial} \stackrel{\text{def}}{=} \partial_{\bar{z}} \stackrel{\text{def}}{=} \frac{1}{2}(\partial_1 + i\partial_2).$$

E.2.2. *Lemma.* (a) $\partial z = 1, \quad \bar\partial \bar z = 1.$

(b) $\bar\partial$, ∂ kill respectively holomorphic and anti-holomorphic functions.

(c) One has $\partial_1 = \partial + \bar\partial$ and $\partial_2 = (\bar\partial - \partial)/i = i(\partial - \bar\partial)$.

E.2.3. *Metrics.* We will extend a metric g to a $\mathbb{C}$-bilinear form on the complexification $(T\Sigma)_{\mathbb{C}}$ and denote

$$g_{zz} \stackrel{\text{def}}{=} g(\partial, \partial), \quad g^{zz} \stackrel{\text{def}}{=} g(dz, dz), \quad etc.$$

Lemma. The transition formulas for metrics g and g^{-1} are are

$$(1)\ \begin{pmatrix} g_{zz} & g_{z\bar z} \\ g_{\bar z z} & g_{\bar z \bar z} \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} g_{11} - g_{22} - 2i\cdot g_{12} & g_{11} + g_{22} \\ g_{11} + g_{22} & g_{11} - g_{22} + 2i\cdot g_{12} \end{pmatrix},$$

$$(2)\ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 2g_{z\bar z} + g_{zz} + g_{\bar z\bar z} & i(g_{zz} - g_{\bar z\bar z}) \\ i(g_{zz} - g_{\bar z\bar z}) & 2g_{z\bar z} - g_{zz} - g_{\bar z\bar z} \end{pmatrix},$$

$$(3)\ \begin{pmatrix} g^{zz} & g^{z\bar z} \\ g^{\bar z z} & g^{\bar z\bar z} \end{pmatrix} = \begin{pmatrix} g^{11} - g^{22} + 2i\cdot g^{12} & g^{11} + g^{22} \\ g^{11} + g^{22} & g^{11} - g^{22} - 2i\cdot g^{12} \end{pmatrix},$$

$$(4)\ \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 2g^{z\bar z} + g^{zz} + g^{\bar z\bar z} & -i(g^{zz} - g^{\bar z\bar z}) \\ -i(g^{zz} - g^{\bar z\bar z}) & 2g^{z\bar z} - g^{zz} - g^{\bar z\bar z} \end{pmatrix}.$$

Proof. (1) We calculate slowly,

$$g_{zz} = g(\partial, \partial) = \tfrac{1}{4}g(\partial_1 - i\partial_2, \partial_1 - i\partial_2) = \tfrac{1}{4}(g_{11} - g_{22} - 2i\cdot g_{12}) = \overline{g_{\bar z\bar z}},$$

$$g_{z\bar z} = g(\partial, \bar\partial) = \tfrac{1}{4}g(\partial_1 - i\partial_2, \partial_1 + i\partial_2) = \tfrac{1}{4}(g_{11} + g_{22}).$$

In the opposite direction, $\frac{1}{2}(g_{11} - g_{22}) = g_{zz} + g_{\bar z\bar z}$ and $g_{12} = \frac{1}{i}(g_{\bar z\bar z} - g_{zz})$.

(3) Similarly,

$$g^{zz} = g^{-1}(dz, dz) = g(d\tau^1 + id\tau^2, d\tau^1 + id\tau^2) = g^{11} - g^{22} + 2i\cdot g^{12} = \overline{g^{\bar z\bar z}} \text{ and}$$
$$g^{z\bar z} = g(dz, d\bar z) = g(d\tau^1 + id\tau^2, d\tau^1 - id\tau^2) = g^{11} + g^{22}.$$

This inverts to $g^{11} - g^{22} = \frac{1}{2}(g^{zz} + g^{\bar z\bar z})$ and $g^{12} = \frac{1}{4i}(g^{zz} - g^{\bar z\bar z})$.

Remarks. We will redo the above coordinate change formulas completely in matrices.

- **(1)** The coordinates and differentials transform by $A = \left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$ and $A^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right)$,
$$\left(\begin{smallmatrix} z \\ \bar z \end{smallmatrix}\right) = A\left(\begin{smallmatrix} \tau^1 \\ \tau^2 \end{smallmatrix}\right), \quad \text{i.e.,} \quad \left(\begin{smallmatrix} \tau^1 \\ \tau^2 \end{smallmatrix}\right) = A^{-1}\left(\begin{smallmatrix} z \\ \bar z \end{smallmatrix}\right)$$
Observe that $\frac{1}{\sqrt{2}}A$ is a unitary matrix, i.e., $A^* \stackrel{\text{def}}{=} \overline{A^{-tr}} = \frac{1}{2}A$.
- **(2)** The dual frames of vector fields use $A^{tr} = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ and $A^{-tr} = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right) = \frac{1}{2}\overline{A}$,
$$\left(\begin{smallmatrix} \partial \\ \bar\partial \end{smallmatrix}\right) = A^{-tr}\left(\begin{smallmatrix} \partial_1 \\ \partial_2 \end{smallmatrix}\right) \quad \text{and} \quad \left(\begin{smallmatrix} \partial_1 \\ \partial_2 \end{smallmatrix}\right) = A^{tr}\left(\begin{smallmatrix} \partial \\ \bar\partial \end{smallmatrix}\right)$$
- **(3)** The matrix of a metric g transforms by
$\left(\begin{smallmatrix} g_{zz} & g_{z\bar z} \\ g_{\bar z z} & g_{\bar z\bar z} \end{smallmatrix}\right) = A^{-tr}\cdot\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right)\cdot A^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right)\cdot\frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right) = \frac{1}{4}\left(\begin{smallmatrix} g_{11} - ig_{21} & g_{12} - ig_{22} \\ g_{11} + ig_{21} & g_{12} + ig_{22} \end{smallmatrix}\right)$.

- **(4)** The dual quadratic form g^{-1} transforms by
$\begin{pmatrix} g^{zz} & g^{z\bar z} \\ g^{\bar z z} & g^{\bar z\bar z}\end{pmatrix} = A\cdot \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix}\cdot A^{tr} = \begin{pmatrix} 1 & i \\ 1 & -i\end{pmatrix}\cdot \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix}\cdot \begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix} = \begin{pmatrix} g^{11}+ig^{21} & g^{12}+ig^{22} \\ g^{11}-ig^{21} & g^{12}-ig^{22}\end{pmatrix}.$

For (3) we use $\begin{pmatrix}\partial \\ \bar\partial\end{pmatrix} = \phi\begin{pmatrix}\partial_1 \\ \partial_2\end{pmatrix}$, for $\phi = A^{-tr}$. From (3) to (4) we replace ϕA^{-tr} with $\phi^{-tr} = A$. Then, $A^* = \frac{1}{2}\overline{A}$ gives $\phi^* = 2\phi$, hence $\phi^{-tr} = 2\overline{\phi}$, so compared with the preceding calculation, all coefficients will be multiplied by 4 and conjugated.

E.3. **The case of conformally flat complex coordinates.** We will compute the stress tensor $\boldsymbol{T} = \boldsymbol{T}[x, g]$ in a holomorphic coordinate conformally flat to g. In particular, at the criticality in x it will reduce to one holomorphic function $\boldsymbol{T}(z) \stackrel{\text{def}}{=} \boldsymbol{T}(\partial, \partial)$.

We start with a Riemannian metric g on an oriented surface (Σ, or). It gives a complex structure $\mathcal{C}$ on Σ, and we use any local holomorphic coordinate z. In this "conformally flat coordinate" g has a very simple form: it is a multiple of the z-standard metric g_{st} (the pull back of the standard metric on $\mathbb{C} = \mathbb{R}^2$), i.e., g is conformal to g_{st}. This will give a simple setting for g-calculations. We use $z, \bar z$ and the real coordinates $\tau^1 = Re(z),\ \tau^2 = Im(z)$.

E.3.1. *Lemma.* (a) Metrics g and (the z-pull-back of) g_{st} are conformal: $g = \omega\cdot g_{st}$.

(b) In real coordinates the metric is given by a scalar matrix and in complex coordinates ∂ and $\bar\partial$ are isotropic

$$g_{ij} = \delta_{ij}\omega, \quad g^{ij} = \delta_{ij}\omega^{-1}, \quad \omega = g_{ii}.$$

$$\begin{pmatrix} g_{zz} & g_{z\bar z} \\ g_{\bar z z} & g_{\bar z\bar z}\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\omega \\ \frac{1}{2}\omega & 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g^{zz} & g^{z\bar z} \\ g^{\bar z z} & g^{\bar z\bar z}\end{pmatrix} = \begin{pmatrix} 0 & 2/\omega \\ 2/\omega & 0\end{pmatrix}, \quad \omega = 2g_{z\bar z} = 2g(\partial, \bar\partial).$$

(c) The Laplacian is $\nabla_g^2 \stackrel{\text{def}}{=} g^{ij}\partial_i\partial_j = \frac{1}{\omega}\partial\bar\partial$

(d) The velocity is $(dx)^2_{G,g} = \frac{4}{\omega}\ \partial x \underset{G}{\cdot} \bar\partial x$.

Proof. (a) The complex structures associated to g and g_{st} coincide! (b) If $g = g_{st}$ the formulas for g_{ij} are clear, and then the general case follows. Now, the formulas with $z, \bar z$ follow using the transition matrices above

$$g_{z\bar z} = g_{\bar z z} = \frac{1}{4}(g_{11} + g_{22}) = \frac{1}{2}\omega, \quad g_{zz} = g_{\bar z\bar z} = \frac{1}{4}(g_{11} - g_{22} - 2ig_{12}) = 0.$$

(c) $\nabla_g^2 \stackrel{\text{def}}{=} g^{ij}\partial_i\partial_j = \frac{1}{\omega}({\partial_1}^2 + {\partial_2}^2) = \frac{1}{4\omega}(\partial + \bar\partial)^2 - \frac{1}{4\omega}(\partial - \bar\partial)^2 = \frac{1}{\omega}\partial\bar\partial$.

(d) $(dx)^2_{G,g} = \partial_i x^\mu\cdot\partial_j x^\nu\cdot G_{\mu\nu}g^{ij} = \omega^{-1}\cdot\partial_i x^\mu\cdot\partial_i x^\nu\cdot G_{\mu\nu}$. So, $\omega\cdot(dx)^2_{G,g}$ equals

$[(\partial+\bar\partial)x^\mu\cdot(\partial+\bar\partial)x^\nu\cdot + i(\partial-\bar\partial)x^\mu\cdot i(\partial-\bar\partial)x^\nu]\, G_{\mu\nu} = [2\partial x^\mu\cdot\bar\partial x^\nu + 2\partial x^\nu\cdot\bar\partial x^\mu]\, G_{\mu\nu} = 4\,\partial x^\mu\cdot\bar\partial x^\nu\cdot G_{\mu\nu}$.

E.3.2. *Corollary.* For a given Euclidean metric g on a surface Σ, locally in Σ we can choose coordinates so that g is conformal to the Euclidean metric $g_{ij} = \delta_{ij}$.

Proof. The Complex Variables proof. This is really the lemma E.3.1.a.

We will also give a *Differential Geometry proof* . (1) Locally, one can find a function $\omega > 0$ such that ωg has Ricci tensor zero.

(2)Since $\dim(\Sigma) = 2$, the vanishing of the Ricci tensor implies the same for the curvature (Riemann tensor). (Curvature R is determined by $R(\partial_1, \partial_2)$, i.e., by $R(\partial_1, \partial_2)\partial_\psi = R^q_{12p}\partial_q$, use the symmetries of the Riemann tensor.)

(3) A flat metric G (i.e., with zero curvature), is "coordinate equivalent" to the unit one (i.e., locally one can choose the coordinates which identify g with the standard metric in $\mathbb{R}^n$).

Part X. Topics

Appendix F. Classical mechanics

F.1. **Symmetries, conserved quantities (momenta) and moment maps.** The main observation is that any conserved quantity, i.e., a function f on $T^*\mathcal{C}$ which is constant on momentum curves, gives a *Poisson* map $T^*\mathcal{C} \xrightarrow{f} \mathbb{R}$. Then one can interpret it as the moment map for $\mathbb{R}$ acting on $T^*\mathcal{C}$ by the vector field $\widehat{f}$ which fixes the Hamiltonian function. So,

Conserved quantities are the same as infinitesimal symmetries.
(and this gives them a moment map interpretation).

Since the conserved quantities have a moment map interpretation, they are also called *momenta*.

Bellow, we notice that this works for any continuous group of symmetries. We follow [**P. Shannahan**].

F.1.1. *Intro.* The standard situation is a group G of symmetries of the configuration space $\mathcal{C}$, so G acts on $\mathcal{C}$ and fixes the equation of motion (Newton's equation). The consequence in the Lagrangian approach is that G acts on $T\mathcal{C}$ and fixes the Lagrangian function. The consequence in the Hamiltonian approach is that G acts on the symplectic manifold $T^*\mathcal{C}$ and fixes the Hamiltonian function. The corresponding moment map $\mu_G : T^*\mathcal{C} \to \mathfrak{g}^*$ has the properties:

(1) Evolution curves are trapped inside the fibers of the moment map μ_G (see Noether's theorem F.1.4).
(2) Therefore, functions on $\mathfrak{g}^*$ are conserved quantities on $T^*\mathcal{C}$.
(3) "Any" conserved quantity produces an infinitesimal symmetry of the system, called the Noether charge.

F.1.2. *Symmetries.* The standard appearance of symmetries is a group G of symmetries of the configuration space $\mathcal{C}$, here G acts on $\mathcal{C}$ and fixes the equation of motion (Newton's equation). The consequence in the Lagrangian approach is that G acts on $T\mathcal{C}$ and fixes the Lagrangian function. The consequence in the Hamiltonian approach is that G acts on the symplectic manifold $T^*\mathcal{C}$ and fixes the Hamiltonian function.

F.1.3. *Conserved quantities.* This are observables that are constant along the evolution curves. For instance:

(1) Momentum is conserved iff there is no force ($F = \frac{dp}{dt}$).
(2) More generally, if q_i does not appear in the Lagrangian L then p_i is a conserved quantity.

One calls the coordinates q_i and p_i conjugate, and says that "p_i is the conjugate momentum to q_I".

F.1.4. *Noether's theorem.*

Theorem. Suppose that the Lagrangian L on TM is invariant under the action of a group G on M. Then for any $\alpha \in \mathfrak{g}$,

$$\langle p(t), \alpha_{q(t)} \rangle$$

is constant.

F.1.5. *Momentum map* $J : TM \to \mathfrak{g}^*$. It is defined using the Lagrangian L on TM. For $v \in T_xM$ and $\alpha \in \mathfrak{g}$,

$$\langle Jv, \alpha \rangle \stackrel{\text{def}}{=} d_v L \ \alpha_x.$$

In terms of the momentum map, Noether's theorem says that Jv is constant along the velocity curves in TM.

F.1.6. *Noether's theorem.* "Any conserved quantity defines an infinitesimal symmetry." More precisely, the conserved quantities given by the moment map form the algebra $\mathcal{O}(\mathfrak{g}^*) = S(\mathfrak{g})$ generated by $\mathfrak{g}$ which does produce infinitesimal symmetries!

F.1.7. *Remark.* Notice that if the typical conserved quantities are the functions on $\mathfrak{g}^*$, then there is a particular kind of "doubly" conserved quantities: the G-invariant functions on $\mathfrak{g}^*$. This should be the kind that is not touched by quantization. (Or we can just mean the functions on the Invariant theory quotient $T^*\mathcal{C}//G$.)

F.1.8. *Summary.* Various kinds of "*momenta*" in physics are related to specific symmetries of particular systems.

The simplest example is when the translation in the direction of the position coordinate q_i preserves the Lagrangian L on TM. This infinitesimal symmetry $\frac{\partial}{\partial q_i}$ produces the moment map which is the same as the associated momentum q_i, which is a conserved quantity.

F.1.9. *The momentum covector associated to the velocity vector* v. This means the canonical 1-form $\pi = \sum p_i \cdot dq_i$ on $T^*\mathcal{C}$, makes it the isomorphism $T^{\mathcal{C}} \to T^*\mathcal{C}$ given by the kinetic energy(?).

F.2. **Relativistic free particle as a constrained system.** We just touch upon the idea of constrained systems.

F.2.1. *The mass-shell constraint.* The size of the canonical momentum (for the length action at least) is constrained by the mass:

$$p^2 + m^2 = 0,$$

see lemma I.3.2.4, or directly: $p^2 = p_\mu p_\nu \eta^{\mu\nu} = \frac{m\dot{x}_\mu}{\sqrt{-dx^2}} \frac{m\dot{x}_\nu}{\sqrt{-dx^2}} \eta^{\mu\nu} = m^2 \cdot \frac{\dot{x}^2}{-\dot{x}^2} = -m^2$.

F.2.2. *Canonical Hamiltonian H_{can} and the Hamiltonian with a constraint H_{const}.* By the canonical Hamiltonian we mean the one given by the standard relation between the Lagrangian and Hamiltonian

$$H_{can} \stackrel{\text{def}}{=} -p\cdot\dot{x} - L.$$

However, in this free particle case it is identically zero since

$$p\cdot\dot{x} = p_\mu \dot{x}^\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\, \dot{x}^\mu = \frac{m\dot{x}^2}{\sqrt{-\dot{x}^2}} = m\sqrt{-\dot{x}^2} = L.$$

The interpretation is that

all dynamics comes from the constraint.

Therefore, we add the mass-shell constraint to the Hamiltonian

$$H_{const} \stackrel{\text{def}}{=} H_{can} + \lambda(p^2 + m^2),$$

here λ is the Lagrange multiplier.

F.2.3. *Canonical formalism.* The Lagrangian L defines the canonical momentum $p = L_{\dot{x}}$ and also the Hamiltonian H. The Poisson structure is then described in terms of the canonical momentum by standard equations

$$\{x^\mu, p_\nu\} \stackrel{\text{def}}{=} \delta_{\mu\nu}.$$

Finally, the evolution is given by the Poisson structure and the Hamiltonian:

$$\dot{x}^\mu = \{x^\mu, H\}.$$

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