

Name: _____

Show all your work and justify all your answers!

1. (25 points) Let A and B be similar $n \times n$ matrices, i.e., assume that there exists an invertible $n \times n$ matrix P , such that $B = P^{-1}AP$. All matrices above are assumed to have entries in the same field F .

- (a) Use the algebraic properties of the determinant to prove that $\det(B) = \det(A)$.
- (b) Prove that $A - \lambda I$ and $B - \lambda I$ are similar matrices, for every scalar λ .
- (c) Prove that the characteristic polynomials of A and B are equal.
- (d) Show that v is an eigenvector of B with eigenvalue λ , if and only if Pv is an eigenvector of A with eigenvalue λ .
- (e) Let $f(x)$ be a polynomial with coefficients in F . Show that $f(A)$ and $f(B)$ are similar matrices. Use it to conclude that $f(A) = 0$, if and only if $f(B) = 0$.
- (f) Prove that the minimal polynomial $m_A(x)$ of A is equal to the minimal polynomial $m_B(x)$ of B .

2. (20 points) Set $A := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

- (a) Find the characteristic polynomial $h(x)$ of A . Show your work!
- (b) Find the minimal polynomial $m(x)$ of A in the polynomial ring $\mathbb{C}[x]$. Do not forget to **carefully** justify your answer!
- (c) Show that A is not similar to a diagonal matrix in $M_3(\mathbb{R})$.
- (d) Find a basis of \mathbb{C}^3 consisting of eigenvectors of A . Hint: Use the notation η for the third root of unity $\cos(2\pi/3) + i\sin(2\pi/3) = (-1 + \sqrt{3}i)/2$. Express your answer in terms of powers of η , in order to simplify the notation and the computations.
- (e) Find an invertible matrix P and a diagonal matrix D , both in $M_3(\mathbb{C})$, such that $P^{-1}AP = D$.

3. (15 points) Factor the polynomial $x^8 - 1$ into its prime factors in $\mathbb{C}[x]$, then in $\mathbb{R}[x]$, and then in $\mathbb{Q}[x]$ (for the latter, you may assume that $\sqrt{2}$ is not a rational number). Prove that each factor you found is prime. Hint: Sketch all the 8-th roots of unity on the unit circle.

4. (20 points) Set $A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$.

- (a) Find the characteristic polynomial $h(x)$ of A .
 - (b) Find the minimal polynomial $m(x)$ of A .
 - (c) Show that the primary decomposition of \mathbb{R}^4 induced by A is a direct sum of two subspaces V_1 and V_2 and find a basis β_i for each V_i , $i = 1, 2$.
 - (d) Let β be the union of the bases β_1 and β_2 you found above. Find the matrix $[A]_\beta$ with respect to the basis β .
5. (20 points) Let $\mathcal{F}(\mathbb{R})$ be the vector space of functions from \mathbb{R} to \mathbb{R} . For each positive integers n , let V_n be the subspace spanned by the set $\beta_n := \{e^t, te^t, \dots, t^n e^t\}$. Let $D : V_n \rightarrow V_n$ be the differentiation operator.
- (a) Let $n = 3$. Find the 4×4 matrix of D in the basis β_3 .
 - (b) Find the minimal polynomial of D (for all n). Prove your answer.
 - (c) Show that the differentiation linear transformation $D : V_n \rightarrow V_n$ is not diagonalizable.