Math 545          Midterm 1          Spring 2012

Name:_____

Show all your work and **justify** all your answer.

1. (20 points) Let $V$ be an $n$ dimensional vector space over $\mathbb{R}$ and let $T : V \to \mathbb{R}^1$ be a non-zero linear transformation from $V$ to the one-dimensional space $\mathbb{R}^1$. Prove that $\dim(\ker(T)) = n - 1$.

2. (20 points) Let $V$ be an inner product space and $w$ a non-zero vector in $V$. Define the reflection $R_w : V \to V$ by

$$R_w(v) \;\; = \;\; v - \frac{2(v, w)}{(w, w)} w. \tag{1}$$

   (a) Prove that the reflection $R_w$ is an orthogonal transformation. You may assume that $R_w$ is a linear transformation.

   (b) Let $C([-\pi, \pi])$ be the vector space of real valued functions defined and continuous on the interval $[-\pi, \pi]$. Endow $C([-\pi, \pi])$ with the inner product $(f, g) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$. Let $V$ be the subspace of $([-\pi, \pi])$ spanned by the set $\beta := \{\cos(x), \cos(2x), \cos(3x)\}$. Recall that $\beta$ is an orthonormal set. Set $w = \cos(x) + \cos(2x) + \cos(3x)$. Let $R_w : V \to V$ be the reflection given by equation (1). Find the matrix $[R_w]_\beta$ with respect to the basis $\beta$ of $V$.

   (c) Keep the notation of part 2b. Explain, without any computations, why the matrix $[R_w]_\beta$ of $R_w$ with respect to the basis $\beta$ of $V$ is an orthogonal matrix. State any theorem you use.

   (d) Check that your matrix in part 2b is indeed an orthogonal matrix.

3. (20 points) Consider $\mathbb{R}^4$ as an inner product space with respect to the dot product. Let $W$ be the subspace of $\mathbb{R}^4$ cut out by the equations

$$x_1 + x_2 + x_3 + x_4 \;\; = \;\; 0, \tag{2}$$
$$x_1 + 2x_2 + x_3 - x_4 \;\; = \;\; 0. \tag{3}$$

   (a) Find a basis for $W$. Justify your answer!

   (b) Use the Gram-Schmidt process and the basis you found in part 3a in order to find an orthonormal basis for $W$.

   (c) Show that the distance from the vector $(0, 1, 2, 1)$ to the subspace $W$ is 2.

4. (20 points) Let $V$ be an $n$ dimensional inner product space.

   (a) Let $\beta = \{u_1, \ldots, u_n\}$ and $\tilde{\beta} := \{\tilde{u}_1, \ldots, \tilde{u}_n\}$ be two orthonormal bases of $V$. Prove that there **exists** a **unique orthogonal transformation** $T : V \to V$ satisfying $T(u_i) = \tilde{u}_i$, for $1 \le i \le n$. Carefully state every theorem you use.

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(b) Let $W$ be a subspace of $V$ of dimension $k$, $0 < k < n$. Prove that every orthonormal basis $\{u_1, \ldots, u_k\}$ of $W$ can be extended to an orthonormal basis of $V$. Hint: Use a property of the Gram-Schmidt process.

(c) Fix a positive integer $k$ satisfying $k < n$. Let $W_1$ and $W_2$ be two subspaces of $V$ of dimension $k$. Prove that there exists an orthogonal transformation $T : V \to V$ such that $T(W_1) = W_2$.

5. (20 points) Denote by $E_{ij}$ the $3 \times 3$ matrix with 1 at the $(i, j)$ entry and zero elsewhere. Then $\beta := \{E_{12}, E_{13}, E_{23}\}$ is a basis of the vector space $U$ of $3 \times 3$ strictly upper triangular matrices. Let $D$ be the vector space of $3 \times 3$ diagonal matrices. Given a diagonal matrix $A$, denote by $T_A : U \to U$ the linear transformation

$$T_A(B) \;=\; AB - BA. \tag{4}$$

(a) Let $A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$. Find the matrix $[T_A]_\beta$ of $T_A$ with respect to the basis $\beta$ of $U$ given above. Express your answer in terms of the diagonal entries $d_1$, $d_2$, $d_3$ of $A$.

(b) Assume that $d_1$, $d_2$, $d_3$ are three *distinct* scalars. Prove that the linear transformation $T_A$ is invertible, where $A$ is the diagonal matrix given in part 5a.

(c) Let $T : D \to L(U, U)$ be the linear transformation sending a diagonal matrix $A$ to the linear transformation $T_A$ given in equation (4), so that $T(A) = T_A$. Find the dimensions of the kernel and image of $T$. Carefully justify your answer.

(d) Show that there exists a diagonal matrix $A$, as in part 5b, such that $T_A$ is invertible, but the inverse $(T_A)^{-1}$ is **not** of the form $T_B$, for any diagonal matrix $B$. Hint: Use part 5c.