Math 545 Solution of Final Exam Spring 2007

1. Let V be a finite dimensional vector space over $\mathbb{C}$ and $T : V \to V$ a linear transformation, such that $T^r = 1$, for some positive integer r. Prove that T is diagonalizable.

   Answer: Let $f(x) = x^r - 1$. Then $f(T) = 0$. Hence, the minimal polynomial $m(x)$ of T divides $f(x)$. Now $f(x) = \prod_{k=0}^{r}(x - \xi^k)$, where $\xi := \cos(2\pi/r) + i\sin(2\pi/r)$ is a primitive root of unity. Hence, $f(x)$ is a product of distinct linear monic terms. Thus, so is any polynomial of positive degree dividing $f(x)$, and in particular, so is $m(x)$. By a theorem, T is diagonalizable over a field F, if and only if its minimal polynomial $m(x)$ factors as a product of distinct monic linear terms in $F[x]$.

2. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D, both with entries in $\mathbb{C}$, such that $P^{-1}AP = D$.

   Answer: The characteristic polynomial is x^2+x+1. The two complex eigenvalues are $\lambda_1 = -(1/2) + i\sqrt{3}/2$ and $\lambda_2 = -(1/2) - i\sqrt{3}/2$. Take P to be the matrix $\begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2}i & \frac{1}{2} - \frac{\sqrt{3}}{2}i \\ 1 & 1 \end{pmatrix}$, so that its first column is a λ_1 eigenvector and its second column is a λ_2 eigenvector. Then $P^{-1}AP = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{3}}{2}i & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix}$.

3. (a) Find an orthonormal basis of $\mathbb{R}^2$, which exhibits the principal axes of the quadratic form $Q(x, y) = 17x^2 + 12xy + 8y^2$.

   Answer: $Q(x, y) = (x, y)S\begin{pmatrix} x \\ y \end{pmatrix}$, where S is the symmetric matrix $\begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix}$. The characteristic polynomial of S is $(x - 5)(x - 20)$. The principal axes are the eigenlines of S. The vector $u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a unit 20-eigenvector and the vector $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is a unit 5-eigenvector. The two vectors are orthogonal, as expected by the Principal Axis Theorem (31.9) in the text, and so $\{u_1, u_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

   (b) Find the matrix P of a *rotation* of $\mathbb{R}^2$, and a diagonal matrix D, such that $Q(x, y) = (x, y)PD(P^t)\begin{pmatrix} x \\ y \end{pmatrix}$. Explain why the P you found is a matrix of a rotation and why the above equality holds.

   Answer: Take $P = (u_1 u_2) = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$. Then P is an orthogonal matrix with $\det(P) = 1$, and is thus the matrix of a rotation. Now $P^tSP = P^{-1}SP = \begin{pmatrix} 20 & 0 \\ 0 & 5 \end{pmatrix} =: D$. So $S = PDP^t$ and $Q(x, y) = (x, y)S\begin{pmatrix} x \\ y \end{pmatrix} = (x, y)PD(P^t)\begin{pmatrix} x \\ y \end{pmatrix}$.

   (c) Use your work above to sketch the graph of $17x^2 + 12xy + 8y^2 = 5$, clearly indicating the principal axes and the coordinates of their points of intersection with the graph.

Answer: Set $\beta := \{u_1, u_2\}$ and consider $\tilde{x}$ and $\tilde{y}$ as the β-coordinates of the vector $\tilde{x}u_1 + \tilde{y}u_2 = P\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$. Now draw in the $\tilde{x}, \tilde{y}$ plane the ellipse $20\tilde{x}^2 + 5\tilde{y}^2 = 5$ and in the x, y plane the ellipse $17x^2 + 12xy + 8y^2 = 5$ and state that the rotation P takes the first ellipse (with the $\tilde{x}$ and $\tilde{y}$ axes as its principal axes) to the second ellipse.

(d) Find an orthogonal (but not orthonormal) basis $\beta = \{v_1, v_2\}$ of $\mathbb{R}^2$, such that the matrix of Q with respect to β is the identity matrix. *Hint: Use your diagonalization in part 3b.*

Answer: Take $\widetilde{P} = P\begin{pmatrix} \frac{1}{\sqrt{20}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}$. Then $\widetilde{P}^t S \widetilde{P} = I$.

4. Parts 4c to 4f below are independent of parts 4a and 4b.

(a) Let u_1 and u_2 be two unit vectors in $\mathbb{R}^3$ and let R_{u_i} be the reflection

$$R_{u_i}(v) = v - 2(u_i, v)u_i$$

of $\mathbb{R}^3$ with respect to the plane $u_i^\perp$ orthogonal to u_i. Prove that the composition $R_{u_2} \circ R_{u_1}$ is a rotation of $\mathbb{R}^3$.

Answer: R_{u_i} is an orthogonal transformation and $\det(R_{u_i}) = -1$. The composition of orthogonal transformations is an orthogonal transformation. Hence, $R_{u_2} \circ R_{u_1}$ is an orthogonal transformation. Its determinant is $\det(R_{u_2} \circ R_{u_1}) = \det(R_{u_2})\det(\circ R_{u_1}) = (-1)^2 = 1$. Hence, $R_{u_2} \circ R_{u_1}$ is a rotation.

(b) Let $u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, and $A := \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Show that $R_{u_2} \circ R_{u_1}$ is equal to multiplication by the matrix A.

Answer: A straight forward calculation.

(c) Find a unit vector v_1, which spans the axis of the rotation of $\mathbb{R}^3$ with matrix A given in part 4b.

Answer: The axis of the rotation is the eigenline with eigenvalue 1. It is spanned by the unit vector $v_1 := \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

(d) Set $v_2 := u_1$, where u_1 is the vector in part 4b. Complete it to an orthonormal basis $\{v_2, v_3\}$ of the plane $v_1^\perp$ orthogonal to the axis of the rotation A.

Answer: A vector v is orthogonal to v_1 and v_2, if and only if it is in the kernel of the matrix $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Such is the unit vector $v_3 := \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

(e) Find the matrix P of a rotation of $\mathbb{R}^3$, whose second column is the vector u_1 in part 4b, such that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$ is in the normal form of the Structure Theorem for Orthogonal Transformations. *Hint: The columns of P should be a suitable orthonormal basis of* $\mathbb{R}^3$ *and* $\det(P) = 1$.

Answer: The matrix $P := (v_1 v_2 v_3) = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$ is orthogonal, of determinant 1, hence the matrix of a rotation. $P(e_1) = v_1$, hence, $P^{-1}AP$ has the desired form, by Theorem (30.5) in the text.

(f) Show that the angle of the rotation A is $\theta = \frac{2\pi}{3}$ or $\theta = \frac{-2\pi}{3}$, depending on the sign of v_1.

Answer: The angle of rotation θ is the angle between v_2 and Av_2 and $\cos(\theta) = (Av_2, v_2) = -(1/2)$. The angle from Av_2 to v_3 is $\pi/2 - \theta$ and $\sin(\theta) = \cos(\pi/2 - \theta) = (Av_2, v_3) = -\sqrt{3}/2$.

5. Find the solution $(y_1(t), y_2(t))$ of the system

$$\begin{array}{rcrcr} \frac{\partial y_1}{\partial t} & = & y_1 & + & y_2 \\ \frac{\partial y_2}{\partial t} & = & -y_1 & + & 3y_2 \end{array}$$

satisfying $y_1(0) = 0$ and $y_2(0) = 1$. *Hint: The matrix A of the system satisfies* $P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, *where* $P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

Answer: The solution of the system of ordinary differential is $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{tA}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now $A = P\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}P^{-1} = P(2I + N)P^{-1}$, where $N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Hence,

$$e^{tA} = e^{tP(2I+N)P^{-1}} = Pe^{t(2I+N)}P^{-1} = P(e^{2tI}e^{tN})P^{-1}.$$

Now $e^{tI} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}$ and $e^{tN} = I + tN + \frac{1}{2}t^2N^2 + \cdots$. The power N^d vanishes, for $d > 1$, and so $e^{tN} = I + tN = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Multiplying out we get $e^{tA} = e^{2t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$ and the final solution is: $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} te^{2t} \\ (1+t)e^{2t} \end{pmatrix}$.

6. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ -7 & -1 & 4 \\ -2 & -1 & 3 \end{pmatrix}$ and work over the field $\mathbb{R}$ of real numbers.

(a) Show that the characteristic polynomial of A is $(x-1)^2(x-2)$.

(b) Find a basis for each eigenspace of A.

Answer: $v_1 := \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ spans the 1-eigenspace and $v_2 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ spans the 2-eigenspace.

(c) Check that each vector you found in part 6b is indeed an eigenvector!

Answer: Chech that $Av_1 = v_1$ and $Av_2 = 2v_2$.

(d) Find the minimal polynomial of A. Justify your answer!

Answer: If the characteristic polynomial is the product $p_1(x)^{d_1} \cdots p_r(x)^{d_r}$, with $\{p_1, \ldots, p_r\}$ distinct monic prime polynomials, then $m(x) = p_1(x)^{e_1} \cdots p_r(x)^{e_r}$, where $1 \leq e_i$ and e_i is the minimal positive integer e, such that

$$\dim\left(\ker[p_i(A)^e]\right) = d_i \deg(p_i(x)),$$

as the right hand side above is the dimension of the direct summand $V_i := \ker\left(p_i(A)^{e_i}\right)$, in the Primary Decomposition Theorem, by the Triangular Form Theorem.

Set $p_1(x) = x - 1$ and $p_2(x) = x - 2$. Then $1 \leq e_2 \leq d_2 = 1$, so $e_2 = 1$. Now $1 \leq e_1 \leq d_1 = 2$. In addition, $\dim\ker(p_1(A)) = \dim\ker(A - I) = 1 < d_1$. Hence, $e_1 = 2$ and

$$m(x) = (x-1)^2(x-2).$$

(e) Find a basis for each V_i in the Primary Decomposition $\mathbb{R}^3 = V_1 \oplus V_2$ with respect to A.

Answer: $V_1 = \ker(p_1(A)^{e_1}) = \ker\left((A-I)^2\right)$ is spanned by $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$V_2 = \ker(p_2(A)^{e_2}) = \ker(A - 2I)$ is spanned by v_2 given in part 6b above.

(f) Find the elementary divisors of A. Carefully justify your answer!

Answer: We need to find a subdecomposition, of each summand V_1, V_2, in the primary decomposition, into a direct sum of cyclic subspaces with respect to A. If $V_i = \langle u_1 \rangle \oplus \langle u_2 \rangle \oplus \cdots$, then the orders $m_{u_1}(x)$, $m_{u_2}(x)$, $\ldots$, are elementary divisors of A.

Now V_2 is one dimensional and is hence cyclic. Thus, $p_2(x) = (x - 2)$ is an elementary divisor.

Consider next the two dimensional V_1. Any vector w in V_1, which is not a 1-eigenvector, will have the property that $\{w, Aw\}$ are linearly independent. Such a w exists, since V_2 is two-dimensional and the 1-eigenspace is one-dimensional. Hence, $V_1 = \langle w \rangle$ is cyclic. The order $m_w(x)$ of w is the minimal polynomial of the restriction $A_{\langle w \rangle}$ of A to $\langle w \rangle$, i.e., to V_1. We have seen that the latter is $(x-1)^2$. Hence, $p_1(x)^{e_1} = (x-1)^2$ happens to be an elementary divisor as well.

Remark: The product of the elementary divisors is always equal to the characteristic polynomial. A homework problems was assigned, titled "continuation of problem 8 in section 25, page 226". In that homework problem you proved that the summands in the primary decomposition are all cyclic, if and only if the minimal polynomial $m(x)$ is equal to the characteristic polynomial. In this case, the elementary divisors are simply the maximal prime powers which divide the minimal polynomial. However, you were expected to provide here a complete and more elementary justification.

(g) Find the Jordan canonical form of A.

Answer: The elementary divisors $p_1(x)^{e_1} := (x-1)^2$ and $p_2(x)^{e_2} = (x-2)^1$ of A determine its Jordan Canonical Form (Theorem 25.16 and Definition 25.17). An elementary divisor of the form $(x - \lambda)^e$ contributes to the Jordan canonical form an $e \times e$ block with λ_i in all its diagonal entries and 1 in all the entries immediately above the diagonal (see bottom of page 224 in the text). Thus, our Jordan canonical form is: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(h) Find an invertible matrix P, such that $P^{-1}AP$ is in Jordan canonical form. Describe your method in complete sentences! Credit will not be given to a solution found by trial and error.

Answer: We follow the procedure of Lemma 25.12, in the special case that all $d_i = 1$ (the prime polynomials are all of degree 1, but powers e_i may be positive). When $d = 1$ and $\langle w \rangle$ is cyclic with elementary divisor $(x - \lambda)^e$, then we choose in Lemma 25.12 for $\langle w \rangle$ the basis

$$\left\{(A - \lambda I)^{e-1}w, (A - \lambda I)^{e-2}w, \cdots, (A - \lambda I)^1 w, w\right\}$$

consisting of e vectors with this specific order.

In our case we can choose the cyclic vector $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (w is in V_1, but w is not a 1-eigenvector). Then $(A - I)w = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$. For V_2 we choose the 2-eigenvector v_2. Take

$$P = ((A - I)w \ w \ v_2) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Then $P^{-1}AP$ is in the Jordan canonical form in part 6g.