Math 545 Midterm 1 Spring 2007

Name:__________________________

1. (10 points) Let V be an n-dimensional vector space over the field $\mathbb{R}$ and let $T : V \to \mathbb{R}^2$ be a linear transformation from V to $\mathbb{R}^2$. Prove that if T is not the zero transformation and T is not onto, then $\dim(null(T)) = n - 1$, where $null(T) := \{v \in V \ : \ T(v) = 0\}$.

2. (10 points) Determine whether there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, satisfying $T(1,1,1) = (1,2)$, $T(1,2,1) = (1,1)$, and $T(2,1,2) = (2,1)$. Justify your answer!

3. (20 points) Let V be the vector space of all polynomial functions
$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$
of degree ≤ 3 with real coefficients c_i, and $T : V \to V$ the linear transformation
$$T(f) \quad = \quad (x+1)\frac{\partial f}{\partial x} - f$$
sending f to $(x+1)$ times its derivative minus f itself.
   (a) Find the matrix $[T]_\beta$ in the basis $\beta = \{1, x, x^2, x^3\}$ of V.
   (b) Find a basis for the null space $null(T) := \{f \ : \ T(f) = 0\}$. Justify your answer!
   (c) Determine the rank of T.
   (d) Find a basis for the image $T(V)$ of T (consisting of polynomials!!!).

4. (20 points) Let V be a finite dimensional vector space over the real numbers, with an inner product. Recall that a linear transformation $T : V \to V$ is called an *orthogonal transformation*, if it preserves length, i.e., $\|T(v)\| = \|v\|$, for all $v \in V$.
   (a) Prove that the product TS, of two orthogonal transformations T and S, is an orthogonal transformation.
   (b) Let T be an orthogonal transformation of V. Show that $\det(T)$ is equal to 1 or -1.

5. (20 points) Let $v_1 = (1,1,0)$, $v_2 = (1,0,1)$, and $v_3 = (1,1,1)$.
   (a) Use the Gram-Schmidt process, and the above basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$, to find an orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$, such that $\text{span}\{v_1, \ldots, v_r\} = \text{span}\{u_1, \ldots, u_r\}$, for $1 \leq r \leq 3$.
   (b) State the definition of an orthonormal basis, and check that the basis you found in part 5a is orthonormal.
   (c) Find the distance from the vector v_3 to the plane spanned by $\{v_1, v_2\}$. (these vectors are given at the beginning of problem 5).
   (d) Explain how to read, from the orthonormal basis you found in part 5a, without any further computations, the equation of the plane spanned by $\{v_1, v_2\}$.

6. (20 points) Let V be an n-dimensional vector space with an inner product and u a unit vector in V (so that $(u,u)=1$). Let $u^\perp$ be the subspace $\{v \in V \ : \ (v,u)=0\}$, orthogonal to u. Recall that the reflection $R_u : V \to V$, of V with respect to $u^\perp$, is given by

$$R_u(v) \ = \ v - 2(v,u)u.$$

   (a) Prove that R_u is a *linear* transformation (it is also easy to show that R_u is an orthogonal transformation, but you are not asked to show it).

   (b) Let u_1 and u_2 be two unit vectors in V. Show that if $(u_1,u_2) = 0$, then $R_{u_1}R_{u_2} = R_{u_2}R_{u_1}$. In other words, the two reflections commute, if the two unit vectors are orthogonal.

   (c) Let $V = \mathbb{R}^2$, with the standard inner product (the dot product), and set $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Find the matrix $[R_u]_\beta$, of the reflection R_u, with respect to the basis $\beta = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\ (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$. Justify your answer!