Math 545 Final Exam Spring 2007

Name:__________________________

Show all your work and justify all your answers!!!

1. (12 points) Let V be a finite dimensional vector space over $\mathbb{C}$ and $T : V \to V$ a linear transformation, such that $T^r = 1$, for some positive integer r. Prove that T is diagonalizable.

2. (12 points) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D, both with entries in $\mathbb{C}$, such that $P^{-1}AP = D$.

3. (18 points)

   (a) Find an orthonormal basis of $\mathbb{R}^2$, which exhibit the principal axes of the quadratic form $Q(x,y) = 17x^2 + 12xy + 8y^2$.

   (b) Find the matrix P of a *rotation* of $\mathbb{R}^2$, and a diagonal matrix D, such that $Q(x,y) = (x,y)PD(P^t)\begin{pmatrix} x \\ y \end{pmatrix}$. Explain why the P you found is a matrix of a rotation and why the above equality holds.

   (c) Use your work above to sketch the graph of $17x^2 + 12xy + 8y^2 = 5$, clearly indicating the principal axes and the coordinates of their points of intersection with the graph.

   (d) Find an orthogonal (but not orthonormal) basis $\beta = \{v_1, v_2\}$ of $\mathbb{R}^2$, such that the matrix of Q with respect to β is the identity matrix. *Hint: Use your diagonalization in part 3b.*

4. (22 points) Parts 4c to 4f below are independent of parts 4a and 4b.

   (a) Let u_1 and u_2 be two unit vectors in $\mathbb{R}^3$ and let R_{u_i} be the reflection

   $$R_{u_i}(v) = v - 2(u_i, v)u_i$$

   of $\mathbb{R}^3$ with respect to the plane $u_i^\perp$ orthogonal to u_i. Prove that the composition $R_{u_2} \circ R_{u_1}$ is a rotation of $\mathbb{R}^3$.

   (b) Let $u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, and $A := \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Show that $R_{u_2} \circ R_{u_1}$ is equal to multiplication by the matrix A.

   (c) Find a unit vector v_1, which spans the axis of the rotation of $\mathbb{R}^3$ with matrix A given in part 4b.

   (d) Set $v_2 := u_1$, where u_1 is the vector in part 4b. Complete it to an orthonormal basis $\{v_2, v_3\}$ of the plane $v_1^\perp$ orthogonal to the axis of the rotation A.

   (e) Find the matrix P of a rotation of $\mathbb{R}^3$, whose second column is the vector u_1 in part 4b, such that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$ is in the normal form of the Structure Theorem for Orthogonal Transformations. *Hint: The columns of P should be a suitable orthonormal basis of $\mathbb{R}^3$ and $\det(P) = 1$.*

(f) Show that the angle of the rotation A is $\theta = \frac{-2\pi}{3}$.

5. (16 points) Find the solution $(y_1(t), y_2(t))$ of the system

$$\begin{array}{rcrcr} \frac{\partial y_1}{\partial t} & = & y_1 & + & y_2 \\ \frac{\partial y_2}{\partial t} & = & -y_1 & + & 3y_2 \end{array}$$

satisfying $y_1(0) = 0$ and $y_2(0) = 1$. *Hint: The matrix A of the system satisfies* $P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, *where* $P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

6. (20 points) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ -7 & -1 & 4 \\ -2 & -1 & 3 \end{pmatrix}$ and work over the field $\mathbb{R}$ of real numbers.

(a) Show that the characteristic polynomial of A is $(x-1)^2(x-2)$.

(b) Find a basis for each eigenspace of A.

(c) Check that each vector you found in part 6b is indeed an eigenvector!

(d) Find the minimal polynomial of A. Justify your answer!

(e) Find a basis for each V_i in the Primary Decomposition $\mathbb{R}^3 = V_1 \oplus V_2$ with respect to A.

(f) Find the elementary divisors of A. Carefully justify your answer!

(g) Find the Jordan canonical form of A.

(h) Find an invertible matrix P, such that $P^{-1}AP$ is in Jordan canonical form. Describe your method in complete sentences! Credit will not be given to a solution found by trial and error.