Let A be the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ -3 & 1 & 5 \end{pmatrix}$. In this problem you are asked to work out explicitly for the matrix A the construction that is carried out in the proof of the Primary Decomposition Theorem 23.9 page 197 in Curtis's textbook.

1. Show that the characteristic polynomial of A is $h(x) = (x-2)^2(x-5)$.

2. Show that the minimal polynomial $m(x)$ of A is equal to its characteristic polynomial.

3. Set $p_1(x) = (x-2)$, $e_1 = 2$, $p_2(x) = (x-5)$, $e_2 = 1$, so that $m(x) = p_1(x)^{e_1} p_2(x)^{e_2}$. Set $q_1(x) := \frac{m(x)}{p_1(x)^{e^1}} = (x-5)$, and $q_2(x) := \frac{m(x)}{p_2(x)^{e^2}} = (x-2)^2$.
   Find polynomials $a_1(x)$ and $a_2(x)$ satisfying the equation
   $$1 = a_1(x)q_1(x) + a_2(x)q_2(x). \tag{1}$$
   Hint: You can find $a_1(x) = ax + b$ of degree 1 and and $a_2(x) = c$ a constant polynomial. Simply regard the coefficients a, b, and c as variables, and equate coefficients in the polynomial equation (1) to get three linear equations in three unknowns.

4. Set $f_i(x) = a_i(x)q_i(x)$ and compute
   $$E_i := f_i(A),$$
   $i = 1, 2$. Show that $E_1E_2 = 0$. Note that E_1 and E_2 commute, so $E_2E_1 = 0$ as well, and $E_1 + E_2 = I$, by construction. Hence, $(E_i)^2 = E_i$ and
   $$\mathbb{R}^3 = E_1(\mathbb{R}^3) \oplus E_2(\mathbb{R}^3) \tag{2}$$
   is a direct sum decomposition, by Lemma 23.6.

5. Compute a basis $\{v_1\}$ of $\ker(p_1(A))$. Then extend it to a basis $\{v_1, v_2\}$ for $\ker\left(p_1(A)^{e_1}\right)$ and show that the latter kernel is equal to $E_1(\mathbb{R}^3)$.

6. Compute a basis $\{v_3\}$ of $\ker\left(p_2(A)^{e_2}\right)$ and show that the kernel is equal to $E_2(\mathbb{R}^3)$.

7. Observe that the direct sum decomposition displayed in equation (2) is translated via the equalities in parts 5 and 6 to the direct sum decomposition
   $$\mathbb{R}^3 = \ker\left(p_1(A)^{e_1}\right) \oplus \ker\left(p_2(A)^{e_2}\right).$$

8. Set $\beta := \{v_1, v_2, v_3\}$. It is a basis of $\mathbb{R}^3$, by the above direct sum decomposition. Compute the β-matrix $[A]_\beta$ of A. It should be block-diagonal with two blocks, each of which is upper triangular.

Answer: $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $[A]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

The entry of $[A]_\beta$ in the first row and second column depends on your choice of the basis $\{v_1, v_2\}$. As long as its non-zero, your answer is correct.