Math 545 Midterm 2 Fall 2011

Name:________________________

Solve 4 out of the following 5 problems. Indicate below which problem you wish not be graded. If you fail to do so, problem 5 will not be graded.
Please do not grade problem ____.
Show all your work and justify all your answers!!!

1. (25 points) Set $A := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

5 pts

(a) Find the characteristic polynomial $h(x)$ of A. Show your work!

$$\det\begin{pmatrix} -x & 0 & 1 \\ 1 & -x & 0 \\ 0 & 1 & -x \end{pmatrix} = -x(x^2) + 1 = -(x^3 - 1)$$

5 pts

(b) Find the minimal polynomial $m(x)$ of A in the polynomial ring $\mathbb{C}[x]$. Do not forget to **carefully** justify your answer!

$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. <u>Method 1:</u> The set $\{I, A, A^2\}$ is linearly indep, since their first columns form the standard basis of $\mathbb{R}^3$. The set $\{I, A, A^2, \underset{\substack{\shortparallel \\ I}}{A^3}\}$ is linearly dependent. Hence, degree $(m(x)) = 3$. Now $f(x) = x^3 - 1$ satisfies $f(A) = 0$. Hence $m(x) \mid f(x)$. Both $f(x)$ and $m(x)$ are monic of degree 3. Thus $m(x) = f(x)$.

<u>Method 2:</u> $h(x)$ has 3 distinct roots $1, \eta, \eta^2$ in $\mathbb{C}$ and each eigenvalue must be a root of $m(x)$. Hence $h(x)$ divides $m(x)$. But $h(A) = A^3 - I = 0$. Hence $m(x) \mid h(x)$. Thus, $m(x) = h(x)$.

5 pts

(c) Show that A is not similar to a diagonal matrix in $M_3(\mathbb{R})$.

$X^3 - 1 = (X-1)(X^2+X+1)$.

X^2+X+1 is prime in $\mathbb{R}[X]$, since $\Delta = 1-4<0$.
Hence, $m(x)$ does not factor as a product of linear factors in $\mathbb{R}[X]$. By a theorem, $m(x)$ is diagonalizable $\Leftrightarrow$ it factors as a product of linear factors with distinct roots.

5 pts

(d) Find a basis of $\mathbb{C}^3$ consisting of eigenvectors of A. Hint: Use the notation η for the third root of unity $\cos(2\pi/3)+i\sin(2\pi/3)$. Express your answer in terms of powers of η, in order to simplify the notation and the computations.

$$(X^3-1) = (X-1)(X-\eta)(X-\eta^2) \qquad -\tfrac{1}{2} \qquad \sqrt{3}/2$$

1 - eigenspace:

$$A - I = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \ker(A-I) = \operatorname{span}\Big\{\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{v_1}\Big\}$$

η - eigenspace:

$$A - \eta I = \begin{pmatrix} -\eta & 0 & 1 \\ 1 & -\eta & 0 \\ 0 & 1 & -\eta \end{pmatrix} \sim \begin{pmatrix} 1 & -\eta & 0 \\ 0 & -\eta^2 & 1 \\ 0 & 1 & -\eta \end{pmatrix} \sim \begin{pmatrix} 1 & -\eta & 0 \\ 0 & 1 & -\eta \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\eta^2 \\ 0 & 1 & -\eta \\ 0 & 0 & 0 \end{pmatrix}$$

$$\ker(A-\eta I) = \operatorname{span}\Big\{\underbrace{\begin{pmatrix} \eta^2 \\ \eta \\ 1 \end{pmatrix}}_{v_2}\Big\}$$

η^2 - eigenspace:

$$A - \eta^2 I = \begin{pmatrix} -\eta^2 & 0 & 1 \\ 1 & -\eta^2 & 0 \\ 0 & 1 & -\eta^2 \end{pmatrix} \sim \begin{pmatrix} 1 & -\eta^2 & 0 \\ 0 & -\eta & 1 \\ 0 & 1 & -\eta^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\eta \\ 0 & 1 & -\eta^2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\ker(A-\eta^2 I) = \operatorname{span}\Big\{\underbrace{\begin{pmatrix} \eta \\ \eta^2 \\ 1 \end{pmatrix}}_{v_3}\Big\}$$

EigenBasis $\{v_1, v_2, v_3\}$.

5 pts

(e) Find an invertible matrix P and a diagonal matrix D, both in $M_3(\mathbb{C})$, such that $P^{-1}AP = D$.

$$P = (v_1\ v_2\ v_3) = \begin{pmatrix} 1 & \eta^2 & \eta \\ 1 & \eta & \eta^2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^2 \end{pmatrix}$$

2. (25 points) Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product and $T : V \to V$ an orthogonal transformation. Prove that $\det(T)$ is equal to 1 or -1.

By a Theorem we know that T is orthogonal $\Leftrightarrow$ for some orthonormal basis $\mathcal{B}$ $A \overset{def}{:=} [T]_{\mathcal{B}}$ is an orthogonal matrix, i.e., $A^t = A^{-1}$.

Now $\det(T) \overset{\text{by definition}}{:=} \det([T]_{\mathcal{B}}) = \det(A)$.

It remains to show that $\det(A) = 1$, if $A^t = A^{-1}$.

$$\underset{\det(I)=1}{\underset{\|}{\det(A^t A)}} = \underbrace{\det(A^t)}_{=\det(A)} \det(A) = \det(A)\det(A)$$

So $\det(A)^2 = 1$.

So $\det(A) = \pm 1$.

3. (25 points) Let V be an n dimensional vector space over $\mathbb{R}$ with an inner product, where $n \geq 3$. Let u_1 and u_2 be two unit vectors in V satisfying $(u_1, u_2) = 0$. Let $T : V \to V$ be the composition $T = R_{u_1} R_{u_2}$, where R_{u_i} is the reflection of V with respect to the subspace $u_i^{\perp}$ orthogonal to u_i. Recall that R_{u_i} is given by

$$R_{u_i}(v) \quad = \quad v - 2(u_i, v)u_i.$$

(a) Show that R_{u_1} and R_{u_2} commute. In other words, use the assumption $(u_1, u_2) = 0$ to prove the equality $R_{u_1}R_{u_2}(v) = R_{u_2}R_{u_1}(v)$, for all v in V.

$$Ru_1 Ru_2(v) = Ru_1\left(v - 2(u_2, v)u_2\right) = [v - 2(u_2,v)u_2] - 2\left(u_1, [v - 2(u_2,v)u_2]\right)u_1$$

$$= v - 2(u_2, v)u_2 - 2(u_1, v)u_1 + 4(u_2,v)\underbrace{(u_1,u_2)}_{0}u_1 =$$

$$= v - 2(u_2, v)u_2 - 2(u_1,v)u_1.$$

The above expression is symmetric in u_1, u_2. Interchanging u_1 and u_2 we see that $Ru_2 Ru_1(v) =$ same expression.

Hence, $Ru_1 Ru_2 = Ru_2 Ru_1$.

(b) Show that $T^2 = 1$. Hint: Show first that $R_{u_i}^2 = 1$.

$$Ru_1(Ru_1(v)) = [v - 2(u_1,v)u_1] - 2\left(u_1, [v - 2(u_1,v)u_1]\right)u_1 =$$

$$= \underbrace{v - 2(u_1,v)u_1 - 2(u_1,v)u_1}_{-4(u_1,v)u_1} + 4\underbrace{(u_1,u_1)}_{1}(u_1,v)u_1 = v.$$

Thus, $Ru_1^2 = 1$.

So $T^2 = (Ru_1 Ru_2)(Ru_1 Ru_2) \overset{\text{by part a}}{=} (Ru_1 Ru_1)(Ru_2 Ru_2) = 1 \cdot 1 = 1$

(c) Show that T is diagonalizable.

Let $f(x) = x^2 - 1$. Then $f(T) = 0$. Thus, the minimal polynomial $m(x)$ of T divides $x^2 - 1$. But $x^2 - 1 = (x-1)(x+1)$ is the product of linear terms with distinct roots. Hence, $m(x)$ is one of $(x-1)$, $(x+1)$, or $(x-1)(x+1)$ and is also a product of linear terms with distinct roots. By a Theorem, T is diagonalizable.

(d) Show that $\{u_1, u_2\}$ span the -1 eigenspace of T. Assume that $T(V) = -V$.

$-V = T(V) = R_{u_1} R_{u_2}(V) = V - 2(u_1, V)u_1 - 2(u_2, V)u_2$. Solve for V, to get $V = (u_1, V)u_1 + (u_2, V)u_2$.

Thus, V is a linear combination of u_1 and u_2.

Conversely, $T(u_1) = u_1 - 2\underbrace{(u_1, u_1)}_{1}u_1 - 2\underbrace{(u_2, u_1)}_{0}u_2 = -u_1$

and similarly, $T(u_2) = -u_2$.

Thus, u_1 and u_2 are eigenvalues with eigenvalue -1.

Method 1: (e) Find the characteristic polynomial of T. Justify your answer!

Extend $\{u_1, u_2\}$ to an orthonormal basis $\mathcal{B} = \{u_1, u_2, u_3, \ldots, u_m\}$ of V. For $j > 2$, u_j is orthogonal to both u_1 and u_2. Thus,

$R_{u_1}(u_j) = u_j - 2\underbrace{(u_1, u_j)}_{0}u_1 = u_j$ and similarly, $R_{u_2}(u_j) = u_j$.

Thus, $T(u_j) = R_{u_1}(R_{u_2}(u_j)) = R_{u_1}(u_j) = u_j$, for $j > 2$.

So $u_3, \ldots, u_m$ are eigenvectors of T with eigenvalue 1. Thus,

$$[T]_{\mathcal{B}} = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \text{ is diagonal.}$$

The characteristic poly is thus $h(x) = (x+1)^2(x-1)^{m-2}$.

Method 2: $m(x)$ and $h(x)$ have the same roots. $m(x) = (x-1)(x+1)$. Thus $h(x) = (x+1)^{d_1}(x-1)^{d_2}$. The power d_1 is 2, by part c, and $d_1 + d_2 = \deg(h) = m$. Thus, $d_2 = m-2$.

4. (25 points) Let V be a vector space and $T : V \to V$ an invertible linear transformation.

(a) Show that if α is an eigenvalue of T, then $\alpha \neq 0$ and α^{-1} is an eigenvalue of T^{-1}.

If α is an eigenvalue of T, then there exists a non-zero vector v such that $T(v) = \alpha v$.
If $\alpha = 0$, then $T(v) = 0 \cdot v = 0$ and so v belongs to $\ker(T)$, which implies that $\ker(T) \neq 0$. If T is invertible, then $\ker(T) = 0$ and so $\alpha \neq 0$.
Apply T^{-1} to both sides of $T(v) = \alpha v$ to get $v = T^{-1}(\alpha v) = \alpha T^{-1}(v)$. So

$$T^{-1}(v) = \alpha^{-1} v.$$

Thus, if v is an eigenvector of T with eigenvalue α, then v is an eigenvector " T^{-1} " " α^{-1}.

8 (pts) (b) Show that if T is diagonalizable, then so is T^{-1}.

Assume that T is diagonalizable. Then there exists a basis $\beta = \{v_1, v_2, \dots \}$ of V, such that every v_i is an eigenvector of T. Part (a) implies that each v_i is also an eigenvector of T^{-1}. Thus, β is also a basis consisting of eigenvectors of T^{-1}. Hence, T^{-1} is diagonalizable.

5. (25 points) Let $\mathcal{F}(\mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ with derivatives of all orders and V the subspace spanned by $\{\cos(x), \sin(x), \cos(2x), \sin(2x)\}$. Let $T : V \to V$ be the differentiation operator, $T(f) = f'$.

~5 pts

(a) Show that the matrix $[T]_\beta$ of T in the basis $\beta := \{\cos(x), \sin(x), \cos(2x), \sin(2x)\}$ of V is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$.

$$[T]_\beta = \left([-\sin(x)]_\beta, [\cos(x)]_\beta, [-2\sin(2x)]_\beta, [2\cos(2x)]_\beta \right) =$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} = A$$

(b) Find the characteristic polynomial $h(x)$ of T. Show your work!

$$h(x) = \det([T]_\beta - xI) = \det \begin{pmatrix} -x & 1 & 0 & 0 \\ -1 & -x & 0 & 0 \\ 0 & 0 & -x & 2 \\ 0 & 0 & -2 & -x \end{pmatrix} =$$

$$= (x^2+1)(x^2+4).$$

(c) Find the minimal polynomial $m(x)$ of T. Justify your answer!

Method 1: Working over the field $\mathbb{R}$ (since V is a vector space over $\mathbb{R}$).

$$A^2 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -4 & \\ & & & -4 \end{pmatrix}, \quad A^2+I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$A^2+4I = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \underbrace{(A^2+I)(A^2+4I)}_{h(A)} = 0.$$

We see that $m(x)$ divides $h(x) = (x^2+1)(x^2+4)$.
Now (x^2+1) and (x^2+4) are prime in $\mathbb{R}[x]$. Hence, $m(x)$ is one of (x^2+1), (x^2+4), $(x^2+1)(x^2+4)$.
Since $A^2+4I \neq 0$ and $A^2+I \neq 0$, then $\boxed{m(x) = (x^2+1)(x^2+4)}$.

Method 2: Work over $\mathbb{C}$, and observe that $h(x) = (x-i)(x+i)(x-2i)(x+2i)$ factors into 4 linear factors with distinct roots. ....

(d) Show that the matrix $[T]_\beta$ is diagonalizable in $M_4(\mathbb{C})$, but not in $M_4(\mathbb{R})$.

$A := [T]_B$ is not diagonalizable in $M_4(\mathbb{R})$, since its minimal poly is NOT a product of linear terms in $\mathbb{R}[X]$.

$[T]_B$ is diagonalizable in $M_4(\mathbb{C})$, since $m(x) = (x-i)(x+i)(x-2i)(x+2i)$ is a product of linear factors with distinct roots.

(e) Show that the primary decomposition of V is a direct sum $V = V_1 \oplus V_2$ of two subspaces and find a basis for each of V_1 and V_2 **(consisting of functions in V)**. Note that V is a vector space over $\mathbb{R}$.

$m(x) = p_1(x) \cdot p_2(x)$, where $p_1(x) = x^2 + 1$, $p_2(x) = x^2 + 4$.

The primary decomposition is

$$V = \underbrace{\ker(T^2 + \mathbb{1})}_{V_1} \oplus \underbrace{\ker(T^2 + 4\mathbb{1})}_{V_2}.$$

$$\ker\left([T]_{\mathcal{B}}^2 + I\right) = \ker\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = \operatorname{Span}\{\overset{[\cos(x)]_{\mathcal{B}}\ \|}{e_1}, \overset{[\sin(x)]_{\mathcal{B}}\ \|}{e_2}\}$$

Hence, $\ker(T^2 + \mathbb{1}) = \operatorname{Span}\{\underbrace{\cos(x), \sin(x)}_{\text{basis for } V_1}\}$

$$\ker\left([T]_{\mathcal{B}}^2 + 4I\right) = \operatorname{Span}\{e_3, e_4\}$$

Hence, $\ker(T^2 + 4\mathbb{1}) = \operatorname{Span}\{\underbrace{\cos(2x), \sin(2x)}_{\text{basis for } V_2}\}$