Math 545 Midterm 2 Fall 2011

Solve 4 out of the following 5 problems. **Show all your work and justify all your answers!!!**

1. (25 points) Set $A := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

   (a) Find the characteristic polynomial $h(x)$ of A. Show your work!

   (b) Find the minimal polynomial $m(x)$ of A in the polynomial ring $\mathbb{C}[x]$. Do not forget to **carefully** justify your answer!

   (c) Show that A is not similar to a diagonal matrix in $M_3(\mathbb{R})$.

   (d) Find a basis of $\mathbb{C}^3$ consisting of eigenvectors of A. Hint: Use the notation η for the third root of unity $\cos(2\pi/3) + i\sin(2\pi/3)$. Express your answer in terms of powers of η, in order to simplify the notation and the computations.

   (e) Find an invertible matrix P and a diagonal matrix D, both in $M_3(\mathbb{C})$, such that $P^{-1}AP = D$.

2. (25 points) Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product and $T : V \to V$ an orthogonal transformation. Prove that $\det(T)$ is equal to 1 or -1.

3. (25 points) Let V be an n dimensional vector space over $\mathbb{R}$ with an inner product, where $n \geq 3$. Let u_1 and u_2 be two unit vectors in V satisfying $(u_1, u_2) = 0$. Let $T : V \to V$ be the composition $T = R_{u_1}R_{u_2}$, where R_{u_i} is the reflection of V with respect to the subspace $u_i^{\perp}$ orthogonal to u_i. Recall that R_{u_i} is given by

$$R_{u_i}(v) \quad = \quad v - 2(u_i, v)u_i.$$

   (a) Show that R_{u_1} and R_{u_2} commute. In other words, use the assumption $(u_1, u_2) = 0$ to prove the equality $R_{u_1}R_{u_2}(v) = R_{u_2}R_{u_1}(v)$, for all v in V.

   (b) Show that $T^2 = 1$. Hint: Show first that $R_{u_i}^2 = 1$.

   (c) Show that T is diagonalizable.

   (d) Show that $\{u_1, u_2\}$ span the -1 eigenspace of T.

   (e) Find the characteristic polynomial of T. Justify your answer!

4. (25 points) Let V be a vector space and $T : V \to V$ an invertible linear transformation.

   (a) Show that if α is an eigenvalue of T, then $\alpha \neq 0$ and α^{-1} is an eigenvalue of T^{-1}.

   (b) Show that if T is diagonalizable, then so is T^{-1}.

5. (25 points) Let $\mathcal{F}(\mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ with derivatives of all orders and V the subspace spanned by $\{\cos(x), \sin(x), \cos(2x), \sin(2x)\}$. Let $T : V \to V$ be the differentiation operator, $T(f) = f'$.

   (a) Show that the matrix $[T]_\beta$ of T in the basis $\beta := \{\cos(x), \sin(x), \cos(2x), \sin(2x)\}$ of V is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$.

   (b) Find the characteristic polynomial $h(x)$ of T. Show your work!

   (c) Find the minimal polynomial $m(x)$ of T. Justify your answer!

   (d) Show that the matrix $[T]_\beta$ is diagonalizable in $M_4(\mathbb{C})$, but not in $M_4(\mathbb{R})$.

   (e) Show that the primary decomposition of V is a direct sum $V = V_1 \oplus V_2$ of two subspaces and find a basis for each of V_1 and V_2 **(consisting of functions in V).** Note that V is a vector space over $\mathbb{R}$.