

Solve **five** of the following six problems. Show all your work and justify all your answers.

1. (22 points) Set $A := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$.

- (a) Show that the characteristic polynomial of A is equal to $(x-3)(x^2+3)$. Show your work!
- (b) Find a basis of \mathbb{C}^3 consisting of eigenvectors of A . Hint: Use the notation $\eta = \frac{-1+\sqrt{3}i}{2}$, $\bar{\eta} = \frac{-1-\sqrt{3}i}{2}$ and note that $\eta\bar{\eta} = 1$ and $\eta^3 = 1$ (so $\bar{\eta} = \eta^2$).
- (c) Find an invertible matrix P and a diagonal matrix D , both in $M_3(\mathbb{C})$, such that $P^{-1}AP = D$.

2. (22 points) Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ and work over the field \mathbb{R} of real numbers.

- (a) Show that the characteristic polynomial of A is $(x+1)(x-1)^2$.
- (b) Find a basis for each eigenspace of A .
- (c) Check that each vector you found in part 2b is indeed an eigenvector!
- (d) Find the minimal polynomial of A . Justify your answer!
- (e) Find a basis for each V_i in the Primary Decomposition $\mathbb{R}^3 = V_1 \oplus V_2$ with respect to A .
- (f) Find the elementary divisors of A . Carefully justify your answer!
- (g) Find the Jordan canonical form of A . Justify your answer!
- (h) Find an invertible matrix P , such that $P^{-1}AP$ is in the Jordan canonical form you provided in part 2g. Describe your method in complete sentences! Credit will not be given to a solution found by trial and error.

3. (22 points)

- (a) The matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ satisfies $P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, where $P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Use this information to obtain formulas for the entries of the matrix e^{tA} as functions of t . *State (in words) each algebraic property, of the exponential of a matrix, you use.*
- (b) Use your work in part 3a to show that the solution $(y_1(t), y_2(t))$ of the system

$$\begin{aligned} \frac{\partial y_1}{\partial t} &= 3y_1 + y_2 \\ \frac{\partial y_2}{\partial t} &= -y_1 + y_2 \end{aligned}$$

satisfying $y_1(0) = a$ and $y_2(0) = b$ is

$$\begin{aligned} y_1(t) &= ae^{2t} + (a+b)te^{2t} \\ y_2(t) &= be^{2t} - (a+b)te^{2t}. \end{aligned}$$

4. (22 points) Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

- (a) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation given by $T(v) = Av$. How many direct summands appear in the Primary Decomposition of \mathbb{R}^4 with respect to T ? Justify your answer!

- (b) Show that for every vector v in \mathbb{R}^4 , the order $m_v(x)$ of v with respect to T is a power of $(x - 2)$.
- (c) Find the orders $m_{e_1}(x)$, $m_{e_2}(x)$, $m_{e_3}(x)$, $m_{e_4}(x)$ with respect to T , for the elements of the standard basis of \mathbb{R}^4 . Hint: You may want to use the following equality (you do not need to prove it) $\text{span}\{v, Av, A^2v, \dots\} = \text{span}\{v, (A - 2I)v, (A - 2I)^2v, \dots\}$.
- (d) Use your work in part 4c in order to find a decomposition of \mathbb{R}^4 as a direct sum $\langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \dots \oplus \langle v_k \rangle$ of cyclic subspaces with respect to T , such that $m_{v_i}(x)$ is a power of a prime polynomial in $\mathbb{R}[x]$. Justify your answer!
- (e) Are the matrices A and B similar? Use your work above to justify your answer. B is given at the beginning of Question 4.
5. (22 points)
- (a) Let A be an $n \times n$ matrix with *real* entries and $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ the linear transformation given by multiplication by A . Assume that $\lambda = a + bi$ is an eigenvalue of T . Show that the complex conjugate $\bar{\lambda} = a - bi$ is an eigenvalue of T as well.
- (b) Let V be an inner product space (over \mathbb{R}) and $T : V \rightarrow V$ an orthogonal transformation. Show that if λ is an eigenvalue of T , then $\lambda = 1$ or $\lambda = -1$.
- (c) Assume that in part 5b the dimension of V is *odd*. Show that T has an eigenvector with eigenvalue 1 or -1 . Hint: Use part 5a.
- (d) Consider \mathbb{R}^3 as an inner product space with respect to the dot product and let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an orthogonal transformation. Assume that u is an eigenvector of T and let W be the plane orthogonal to u . Show that W is T -invariant (i.e., that $T(w)$ belongs to W , for all w in W).
- (e) Keep the notation of part 5d. Show that the restriction $T_W : W \rightarrow W$ of T to W is an orthogonal transformation.
- (f) Keep the notation of part 5d. Assume, in addition, that the eigenvalue of u is 1 and that $\det(T) = 1$. Show that there exists a basis $\beta_2 := \{v, w\}$ of W , such that the matrix of T with respect to the basis $\beta := \{u, v, w\}$ is of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$, for some angle θ . Hint: You may use the fact that a 2×2 orthogonal matrix with determinant 1 is the matrix of a rotation of \mathbb{R}^2 .
6. (22 points)
- (a) Recall that a linear transformation $E : V \rightarrow V$ is *idempotent*, if E is non-zero, and $E^2 = E$. Show that every idempotent linear transformation is diagonalizable.
- (b) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation with standard matrix A and minimal polynomial $m(x) = (x - 2)^2(x - 3)$. Set $q_1(x) = (x - 3)$ and $q_2(x) = (x - 2)^2$. Find a polynomial $a_1(x) = ax + b$ of degree 1 and a constant polynomial $a_2(x) = c$, such that $a_1(x)q_1(x) + a_2(x)q_2(x) = 1$ (the constant polynomial 1).
- (c) Set $E_1 := a_1(A)q_1(A)$ and $E_2 := a_2(A)q_2(A)$. Show that $E_1E_2 = E_2E_1$, $E_1 + E_2 = I$, where I is the identity matrix.
- (d) Keep the notation of part 6c. Show that $E_1E_2 = 0$.
- (e) Keep the notation of part 6c. Show that E_1 and E_2 are idempotent matrices.
- (f) Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 3 \end{pmatrix}$. The minimal polynomial of A is $m(x) = (x - 2)^2(x - 3)$. You are **not** asked to prove it. Calculate the matrices E_1 and E_2 for this matrix A . Hint: Start with E_2 to save calculations.