Math 545 Midterm 2 Fall 2010

Name:__________________________

Solve 4 out of the following 5 problems. Indicate below which problem you wish not be graded. If you fail to do so, problem 5 will not be graded.

Please do not grade problem ____.

Show all your work and justify all your answers!!!

1. (25 points) Set $A := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

   (a) Find the characteristic polynomial $h(x)$ of A.

   (b) Find the minimal polynomial $m(x)$ of A in the polynomial ring $\mathbb{C}[x]$. Do not forget to **carefully** justify your answer!

   (c) Show that A is not similar to a diagonal matrix in $M_2(\mathbb{R})$.

   (d) Find a basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

   (e) Find an invertible matrix P and a diagonal matrix D, both in $M_2(\mathbb{C})$, such that $P^{-1}AP = D$.

2. (25 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by multiplication by $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

   (a) Find the characteristic polynomial of T.

   (b) Find the minimal polynomial of T. Justify your answer!

   (c) Determine if T is diagonalizable. Justify your answer!

   (d) Find the eigenvalues of T.

   (e) Find a basis for each eigenspace of T.

   (f) Find an upper triangular matrix B and an invertible matrix P, such that $B = P^{-1}AP$. Carefully explain, in complete sentences, your method for finding P. Credit will not be given for an answer obtained by trial and error.

3. (25 points) Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear transformation satisfying $T^3 = T$. Show that T is diagonalizable.

4. (25 points) Let V be a finite dimensional vector space and $T : V \to V$ a linear transformation.

   (a) Show that if T is invertible, then x does not divide the minimal polynomial $m(x)$ of T.

   (b) Prove that T is invertible, if and only if the constant term a_0 of the minimal polynomial $m(x) = a_0 + a_1x + \ldots + a_kx^k$ of T is different from zero. Show, furthermore, that when T is invertible, then T^{-1} can be expressed as a polynomial in T.

5. (25 points) Let $\mathcal{F}(\mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ with derivatives of all orders and V the subspace spanned by $\{e^x, e^{2x}, xe^x, xe^{2x}\}$. Let $T : V \to V$ be the differentiation operator, $T(f) = f'$.

(a) Show that the matrix $[T]_\beta$ of T in the basis $\beta := \{e^x, e^{2x}, xe^x, xe^{2x}\}$ of V is
$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(b) Find the characteristic polynomial $h(x)$ of T.

(c) Find the minimal polynomial $m(x)$ of T. Justify your answer!

(d) Show that the primary decomposition of V is a direct sum $V = V_1 \oplus V_2$ of two subspaces and find a basis for each of V_1 and V_2 **(consisting of functions in V).**