Math 545 Solution of Midterm 1 Fall 2010

1. (20 points) Let U, V, and W be vector spaces of dimensions m, n, and p, respectively. Let $S : U \to V$ and $T : V \to W$ be linear transformations satisfying $TS = 0$ (the zero linear transformation from U to W). Prove that $\mathrm{rank}(S) + \mathrm{rank}(T) \leq n$. Hint: Recall that $\mathrm{rank}(S) = \dim(im(S))$ and relate the image $im(S)$ to the kernel (i.e., null space) of T.

   Answer: We are given that $T(S(u)) = 0$, for every $u \in U$. Hence, $S(u)$ belongs to $\ker(T)$, for all $u \in U$. We conclude that $im(S)$ is a subspace of $\ker(T)$, and so $\dim(im(S)) \leq \dim(\ker(T))$. The rank nullity theorem yields the left equality below

   $$\dim(V) \quad = \quad \dim(\ker(T)) + \dim(im(T)) \quad \geq \quad \dim(im(S)) + \dim(im(T)),$$

   while the right inequality was proven above. Now $n = \dim(V)$ and the right hand side is $\mathrm{rank}(S) + \mathrm{rank}(T)$, by definition of the *rank* of a linear transformation.

2. (20 points) Let M_3 be the vector space of 3×3 matrices with real coefficients, A the 3×3 matrix with all entries equal 1, and I the 3×3 identity matrix. Determine whether there exists a linear transformation $T : \mathbb{R}^3 \to M_3$, satisfying $T(1,1,1) = I + 2A$, $T(1,2,1) = I + A$, and $T(3,2,1) = 2I + A$.
   Hint: Use a theorem to justify your answer with as few computations as possible!

   Answer: There exists a unique such linear transformation for the following reason. The three vectors $\{(1,1,1), (1,2,1), (3,2,1)\}$ form a basis β of $\mathbb{R}^3$, since they are linearly independent, as is easily seen by row reduction of the 3×3 matrix with these vectors as columns. Theorem (13.1) page 99 states that for every vector space W and every choice of three vectors w_1, w_2, and w_3 in W, there **exists** a unique linear transformation $T : \mathbb{R}^3 \to W$, which maps v_i to w_i, for $1 \leq i \leq 3$. More explicitly, a vector v in $\mathbb{R}^3$ with β-coordinate vector $[v]_\beta = (c_1, c_2, c_3)$ is sent by T to $\sum_{i=1}^{3} c_i w_i$. Apply this theorem with $W = M_3$.

3. (20 points) Let V be the vector space of all polynomial functions

   $$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

   of degree ≤ 3 with real coefficients c_i. Let $T : V \to V$ the linear transformation

   $$T(f) \quad = \quad (x+1)\frac{\partial f}{\partial x} - f,$$

   sending a polynomial f to $(x+1)$ times its derivative minus f itself, and $S : V \to V$ the linear transformation $S(f) \quad = \quad x\dfrac{\partial f}{\partial x} - 2f.$

   (a) Compute the composite linear transformation $TS : V \to V$, i.e., find polynomials $a(x)$, $b(x)$, $c(x)$, such that $T(S(f)) = a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} + c(x) f$.
   Answer: $T(S(f)) = T\left(x\frac{\partial f}{\partial x} - 2f\right) = (x+1)\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x} - 2f\right) - \left[x\frac{\partial f}{\partial x} - 2f\right] = (x^2 + x)\frac{\partial^2 f}{\partial x^2} + (-2x - 1)\frac{\partial f}{\partial x} + 2f$. The last equality uses the product rule for differentiation.

(b) Find the matrix $[S]_\beta$ of S (**not** of TS) in the basis $\beta = \{1, x, x^2, x^3\}$ of V.

Answer: $[S]_\beta = ([S(1)]_\beta [S(x)]_\beta [S(x^2)]_\beta [S(x^3)]_\beta) = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(c) You are given that the matrix of T in the basis β is $[T]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Use this information to find the matrix of TS in the basis β. (Credit will not be given for a solution using another method).

Answer: $[TS]_\beta = [T]_\beta [S]_\beta =$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(d) Find a basis for the kernel $\ker(TS) := \{f \ : \ TS(f) = 0\}$. Justify your answer!

Answer: The kernel of the matrix $[TS]_\beta$ has basis $\left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

These are the β-coordinate vectors of the basis $\{\frac{1}{2} + x, x^2\}$ of TS.

(e) Find a basis for the image $TS(V)$ of TS (consisting of polynomials!!!).

Answer: The pivot columns of $[TS]_\beta$ are the coordinate vectors of the basis $\{2, 3x^2 + 2x^3\}$ of $im(TS)$.

4. (20 points) Let $C([-\pi, \pi])$ be the vector space of continuous real valued functions on the interval $[-\pi, \pi]$. Recall that the pairing $(f, g) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$ defines an inner product on $C([-\pi, \pi])$. Recall also that the set

$$\Sigma := \{\sin(nx) \ : \ n \text{ is a positive integer}\}$$

is an orthonormal subset. You do **not** need to verify this fact.

(a) Let f and g be two functions given by

$$\begin{aligned} f(x) &:= f_1 \sin(x) + f_2 \sin(2x) + f_3 \sin(3x), \\ g(x) &:= g_1 \sin(x) + g_2 \sin(2x) + g_3 \sin(3x), \end{aligned}$$

where the coefficients f_i and g_i, $1 \le i \le 3$, are real numbers. Express the inner product (f, g) in terms of the coefficients f_i and g_i.

Answer: We know that the set $\{\sin(x), \sin(2x), \sin(3x)\}$ is orthonormal, i.e., $(\sin(nx), \sin(mx)) = 0$, if $n \neq m$, and $(\sin(nx), \sin(nx)) = 1$. Hence, $(f, g) = f_1g_1 + f_2g_2 + f_3g_3$.

(b) Set $f(x) = \sin(x) + \sin(3x)$, and $g(x) = \sin(x) + \sin(2x) + \sin(3x)$. Use the Gram-Schmidt process to find an orthonormal basis $\{u_1, u_2\}$ for the subspace W of $C([-\pi, \pi])$ spanned by the set $\{f, g\}$.

Answer: $(f, f) = 2$, so $u_1 := \frac{1}{\sqrt{2}} f = \frac{1}{\sqrt{2}}(\sin(x) + \sin(3x))$. Let $\hat{g}$ be the projection of g to $\text{span}\{u_1\}$. Then $\hat{g} = (g, u_1)u_1 = \sin(x) + \sin(3x)$. Set $\tilde{u}_2 := g - \hat{g} = \sin(2x)$. Then $u_2 := \frac{1}{||\tilde{u}_2||}\tilde{u}_2 = \tilde{u}_2 = \sin(2x)$.

(c) Let h be the function $h(x) = x$. Find the projection $\hat{h}$ of h to W. Hint: Integration by parts yields $\int x\sin(nx)dx = -\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx) + C$. Use it to show that $(x, \sin(nx)) = (-1)^{n+1}\frac{2}{n}$.

Answer: $(x, \sin(nx)) = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{\pi} = \frac{-2}{n}\cos(n\pi) = (-1)^{n+1}\frac{2}{n}$.

We have $\hat{h} = (h, u_1)u_1 + (h, u_2)u_2 = \frac{4}{3}\sin(x) - \sin(2x) + \frac{4}{3}\sin(3x)$.

(d) Find an orthogonal (not necessarily orthonormal) basis $\{u_1, u_2, u_3\}$ of the subspace V of $C([-\pi, \pi])$ spanned by the functions $\{f, g, h\}$ in parts 4b and 4c.

Answer: Take the basis $\{u_1, u_2, h - \hat{h}\} =$
$\left\{\frac{1}{\sqrt{2}}[\sin(x) + \sin(3x)], \sin(2x), x - \frac{4}{3}\sin(x) + \sin(2x) - \frac{4}{3}\sin(3x)\right\}$.

5. (20 points) Let V be an n-dimensional vector space with an inner product, and $W \subset V$ an r-dimensional subspace of V, with $0 < r < n$.

(a) Show that *every* orthonormal basis $\beta_1 := \{u_1, \ldots, u_r\}$ of W can be completed to an orthonormal basis $\beta := \{u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\}$ of V.

Answer: Any basis of W can be completed to a basis of V. Let $\{u_1, \ldots, u_r, v_{r+1}, \ldots, v_n\}$ be a basis of V. Performing the Gram-Schmidt process on this basis does not change the first r vectors, and thus produces an orthonormal basis of V, whose first r elements are $\{u_1, \ldots, u_r\}$.

(b) Let $P : V \to V$ be the orthogonal projection onto W. Find the matrix of P in a basis β of the type described in part 5a.

Answer: The projection is given by the formula

$$P(v) = \sum_{i=1}^{r}(v, u_i)u_i.$$

As the basis β is orthonormal, $P(u_i) = u_i$, for $1 \leq i \leq r$, and $P(u_i) = 0$, for $r + 1 \leq i \leq n$. Hence, the matrix $[P]_\beta$ is diagonal, with the first r diagonal entries all equal 1, and the rest are 0.

(c) Let $I : V \to V$ be the identity linear transformation and P the projection in part 5b. Prove that $I - 2P$ is an orthogonal transformation from V to V.

Answer:

Method I: We have $(I - 2P)(u_i) = \begin{cases} -u_i & \text{if} \quad i \leq r, \\ u_i & \text{if} \quad i \geq r. \end{cases}$

We see that $(I - 2P)$ takes the orthonormal basis $\{u_1, \ldots, u_n\}$ to the orthonormal basis $\{-u_1, \ldots, -u_r, u_{r+1}, \ldots, u_n\}$. Thus $I - 2P$ is orthogonal, by Theorem .15.11 page 127 (characterization 3).

Method II: Set $A := [I - 2P]_\beta = I_n - 2[P]_\beta$. Then A is a diagonal matrix with ± 1 on the diagonal. Hence $A^t = A^{-1}$, and thus $I - 2P$ is orthogonal, by Theorem .15.11 page 127 (characterization 4).