Math 545 Midterm 1 Fall 2010

Name:________________________

1. (20 points) Let U, V, and W be vector spaces of dimensions m, n, and p, respectively. Let $S : U \to V$ and $T : V \to W$ be linear transformations satisfying $TS = 0$ (the zero linear transformation from U to W). Prove that $\text{rank}(S) + \text{rank}(T) \leq n$. Hint: Recall that $\text{rank}(S) = \dim(im(S))$ and relate the image $im(S)$ to the kernel (i.e., null space) of T.

2. (20 points) Let M_3 be the vector space of 3×3 matrices with real coefficients, A the 3×3 matrix with all entries equal 1, and I the 3×3 identity matrix. Determine whether there exists a linear transformation $T : \mathbb{R}^3 \to M_3$, satisfying $T(1,1,1) = I + 2A$, $T(1,2,1) = I + A$, and $T(3,2,1) = 2I + A$.
Hint: Use a theorem to justify your answer with as few computations as possible!

3. (20 points) Let V be the vector space of all polynomial functions

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

of degree ≤ 3 with real coefficients c_i. Let $T : V \to V$ the linear transformation

$$T(f) \quad = \quad (x+1)\frac{\partial f}{\partial x} - f,$$

sending a polynomial f to $(x+1)$ times its derivative minus f itself, and $S : V \to V$ the linear transformation $\quad S(f) \quad = \quad x\dfrac{\partial f}{\partial x} - 2f.$

   (a) Compute the composite linear transformation $TS : V \to V$, i.e., find polynomials $a(x)$, $b(x)$, $c(x)$, such that $T(S(f)) = a(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} + c(x)f$.

   (b) Find the matrix $[S]_\beta$ of S (**not** of TS) in the basis $\beta = \{1, x, x^2, x^3\}$ of V.

   (c) You are given that the matrix of T in the basis β is $[T]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.
   Use this information to find the matrix of TS in the basis β. (Credit will not be given for a solution using another method).

   (d) Find a basis for the kernel $\ker(TS) := \{f \ : \ TS(f) = 0\}$. Justify your answer!

   (e) Find a basis for the image $TS(V)$ of TS (consisting of polynomials!!!).

4. (20 points) Let $C([-\pi, \pi])$ be the vector space of continuous real valued functions on the interval $[-\pi, \pi]$. Recall that the pairing $(f, g) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$ defines an inner product on $C([-\pi, \pi])$. Recall also that the set

$$\Sigma := \{\sin(nx) \ : \ n \text{ is a positive integer}\}$$

is an orthonormal subset. You do **not** need to verify this fact.

(a) Let f and g be two functions given by

$$\begin{aligned} f(x) &:= f_1 \sin(x) + f_2 \sin(2x) + f_3 \sin(3x), \\ g(x) &:= g_1 \sin(x) + g_2 \sin(2x) + g_3 \sin(3x), \end{aligned}$$

where the coefficients f_i and g_i, $1 \leq i \leq 3$, are real numbers. Express the inner product (f, g) in terms of the coefficients f_i and g_i.

(b) Set $f(x) = \sin(x) + \sin(3x)$, and $g(x) = \sin(x) + \sin(2x) + \sin(3x)$. Use the Gram-Schmidt process to find an orthonormal basis $\{u_1, u_2\}$ for the subspace W of $C([-\pi, \pi])$ spanned by the set $\{f, g\}$.

(c) Let h be the function $h(x) = x$. Find the projection of h to W. Hint: Integration by parts yields $\int x \sin(nx) dx = -\frac{x}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) + C$. Use it to show that $(x, \sin(nx)) = (-1)^{n+1} \frac{2}{n}$.

(d) Find an orthogonal (not necessarily orthonormal) basis $\{u_1, u_2, u_3\}$ of the subspace V of $C([-\pi, \pi])$ spanned by the functions $\{f, g, h\}$ in parts 4b and 4c.

5. (20 points) Let V be an n-dimensional vector space with an inner product, and $W \subset V$ an r-dimensional subspace of V, with $0 < r < n$.

(a) Show that *every* orthonormal basis $\beta_1 := \{u_1, \ldots, u_r\}$ of W can be completed to an orthonormal basis $\beta := \{u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\}$ of V.

(b) Let $P : V \to V$ be the orthogonal projection onto W. Find the matrix of P in a basis β of the type described in part 5a.

(c) Let $I : V \to V$ be the identity linear transformation and P the projection in part 5b. Prove that $I - 2P$ is an orthogonal transformation from V to V.