Math 545 Final Exam Fall 2010

Solve the five problems 1, 2, 3, 4, 5, and **only one** out of the last two: 6 **or** 7. If you fail to do so, problem 7 will not be graded.
Please do not grade problem ____.
Show all your work and justify all your answers!!!

1. (20 points) Let V be a 4 dimensional vector space over $\mathbb{R}$ with an inner product, u a unit vector in V, so that $(u,u) = 1$, and let W be the subspace of V orthogonal to u. Recall that the reflection of V with respect to the subspace W is given by the formula
$$R(v) = v - 2(v,u)u.$$
   (a) Let $\{w_1, w_2, w_3\}$ be a basis of W and set $\beta := \{u, w_1, w_2, w_3\}$. Find the matrix of R with respect to the basis β.
   (b) Find the characteristic polynomial of R.
   (c) Find the minimal polynomial of R. Justify all your answers above!

2. (15 points) Let V be an n-dimensional vector space over $\mathbb{C}$ and $T : V \to V$ a linear transformation. Recall that T is *nilpotent*, if $T^k = 0$, for some positive integer k.
   (a) Prove that T is nilpotent, if all its eigenvalues are zero.
   (b) Conversly, prove that if T is nilpotent, then all the eigenvalues are zero.

3. (15 points) Determine the Jordan canonical form (over $\mathbb{C}$) of the three matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Justify your answer with as little computations as possible. For each of the three pairs of matrices, $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$ determine if the two matrices are similar.

4. (20 points)
   (a) The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ satisfies $P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where $P = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Use this information to obtain formulas for the entries of the matrix e^{tA} as functions of t. *State (in words) each algebraic property, of the exponential of a matrix, you use.*
   (b) Use your work in part 4a to find the solution $(y_1(t), y_2(t))$ of the system
$$\begin{array}{rcrcr} \frac{\partial y_1}{\partial t} & = & & & -y_2 \\ \frac{\partial y_2}{\partial t} & = & y_1 & + & 2y_2 \end{array}$$
   satisfying $y_1(0) = 0$ and $y_2(0) = 1$.

5. (20 points) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ -1 & -4 & -1 \end{pmatrix}$ and work over the field $\mathbb{R}$ of real numbers.
   (a) Show that the characteristic polynomial of A is $(x+1)(x-2)^2$.

(b) Find a basis for each eigenspace of A.

(c) Check that each vector you found in part 5b is indeed an eigenvector!

(d) Find the minimal polynomial of A. Justify your answer!

(e) Find a basis for each V_i in the Primary Decomposition $\mathbb{R}^3 = V_1 \oplus V_2$ with respect to A.

(f) Find the elementary divisors of A. Carefully justify your answer!

(g) Find the Jordan canonical form of A.

(h) Find an invertible matrix P, such that $P^{-1}AP$ is in Jordan canonical form. Describe your method in complete sentences! Credit will not be given to a solution found by trial and error.

6. (10 points) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be multiplication by $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and set $e_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Calculate the order $m_{e_3}(x)$ with respect to T. Justify your answer!

7. (10 points)

(a) Find the rational canonical form of a linear transformation over $\mathbb{R}$, whose two elementary divisors are as follows. $\{(x-2)^2,\ x^2+x+1\}$.

(b) Now work over $\mathbb{C}$ and find the elementary divisors and the Jordan canonical form of the linear transformation given in part 7a. Justify your answer!