

MATH 235 SPRING 2011
FINAL EXAM

1. (18 points)

- (a) Consider the complex plane \mathbb{C} as a two dimensional vector space with basis $\beta = \{1, i\}$. Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be multiplication by the complex number $2 + 3i$, i.e., $T(z) = (2 + 3i)z$. Find the β -matrix of T .

(a) $A = (\vec{a}_1, \vec{a}_2)$

$$\begin{array}{ccc} & \xrightarrow{x+iy} & \mathbb{C} \xrightarrow{T} \mathbb{C} \\ \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]_{\mathbb{B}}^{-1} & \downarrow \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]_{\mathbb{B}} & \downarrow \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]_{\mathbb{B}} \\ \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right) & \xrightarrow{\mathbb{R}^2} & \left(\begin{smallmatrix} x+iy \\ y \end{smallmatrix} \right) \xrightarrow{\mathbb{R}^2} A \end{array}$$

A is the standard matrix of the composite linear transformation $\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]_{\mathbb{B}} \circ T \circ \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]_{\mathbb{B}}^{-1}$

$$\vec{a}_1 = A \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]_{\mathbb{B}} \left(T \left(\underbrace{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right]}_{1+0 \cdot i} \right) \right) = \left[\begin{smallmatrix} 2+3i \\ 0+3i \end{smallmatrix} \right]_{\mathbb{B}} = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right)$$

$$\vec{a}_2 = A \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]_{\mathbb{B}} \left(T \left(\underbrace{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right]}_{0+1 \cdot i} \right) \right) = \left[\begin{smallmatrix} 0+2i \\ -3+2i \end{smallmatrix} \right]_{\mathbb{B}} = \left(\begin{smallmatrix} -3 \\ 2 \end{smallmatrix} \right)$$

$$\text{So } A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

- (b) Let $A = \begin{pmatrix} 5 & -5 \\ 4 & 1 \end{pmatrix}$. Find the characteristic polynomial of A and determine the eigenvalues of A .

$$\det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & -5 \\ 4 & 1-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 25$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36 - 100}}{2} = 3 \pm \sqrt{9 - 25} = 3 \pm 4i$$

$$\lambda_1 = 3 + 4i, \quad \lambda_2 = 3 - 4i$$

- (c) Find an invertible matrix P , with complex entries, and a diagonal matrix D , such that $P^{-1}AP = D$. Justify your answer!

Basis for $\lambda_1 = 3+4i$ eigenspace:

$$A - (3+4i)I = \begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2}-i \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} (+\frac{1}{2}+i)z_2 \\ z_2 \end{pmatrix} = z_2 \begin{pmatrix} 1/2+i \\ 1 \end{pmatrix} \text{ choose } z_2 = 2$$

$$v_1 = \begin{pmatrix} 1+2i \\ 2 \end{pmatrix} \text{ spans the } (3+4i)-\text{eigenspace}$$

$$\text{check: } \begin{pmatrix} 5 & -5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1+2i \\ 2 \end{pmatrix} = \begin{pmatrix} -5+10i \\ 6+8i \end{pmatrix} = (3+4i) \begin{pmatrix} 1+2i \\ 2 \end{pmatrix}$$

$v_2 = \bar{v}_1 = \begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$ is a basis for the $3-4i$ eigenspace.

$$P = (v_1 \ v_2) = \begin{pmatrix} (1+2i) & (1-2i) \\ 2 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 3+4i & 0 \\ 0 & 3-4i \end{pmatrix}$$

- (d) Find an invertible matrix S , with real entries, and real numbers a, b , such that

$$S^{-1}AS = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

If we choose $a+bi = 3+4i$ ($a=3, b=4$), then the algorithm in sec 5.5 of our textbook states to take v_2 as an eigenvector with eigenvalue $(3-4i)$ and let $S = (\text{Re}(v_2) \ \text{Im}(v_2))$.

$$\text{In our case } \text{Re}(v_2) = \text{Re}\left(\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}, \text{ check: } \text{Im}(v_2) = \text{Im}\left(\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}. \text{ So}$$

$$S^{-1}AS = \frac{1}{4} \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 & -5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 8 & 2 \\ -6 & 11 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 12 & -16 \\ 16 & 12 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$