Math 235H Midterm 1 Fall 2018

Your Name: ______________________________

Student ID: ______________________________

This is a 90 minutes exam. This exam paper consists of 5 questions. It has 8 pages.

The use of calculators is not allowed on this exam. You may use one letter size page of notes (both sides), but no books.

It is not sufficient to just write the answers. You must *explain* how you arrive at your answers.

1. (20 points) a) Show that the row **reduced** echelon augmented matrix of the system

$$\begin{aligned} x_1 - x_2 + x_4 + x_5 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 + 2x_5 &= 0 \\ -x_1 + x_2 &= 1 \end{aligned} \quad \text{is} \quad \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Use at most five elementary row operations. Clearly write in words each elementary row operation you use.

b) Find the general solution for the system in parametric form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} =$$

c) Find the general solution of the associated homogeneous system (replacing the constant, on the right hand side of the equations in part a, by zeros). Try to avoid computations. Justify your answer!

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} =$$

d) Find two vectors $\{v_1, v_2\}$, such that the solution set in Part c is $\operatorname{span}\{v_1, v_2\}$. Justify your answer starting from the definition of $\operatorname{span}\{v_1, v_2\}$.

2. (20 points) Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 3 \\ 2 \\ h \end{bmatrix}$.

a) For which real numbers h is the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ in $\operatorname{span}\{v_1, v_2, v_3\}$? Justify your answer!

b) For which values of h are the three vectors $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer without additional computations.

3. (20 points)

   (a) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation reflecting each vector with respect to the line $x_1 = x_2$. Find the standard matrix of R.

   (b) Find the standard matrix of the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by $S(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2)$.

   (c) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(\vec{x}) = S(R(\vec{x}))$. Show all your work!

4. (20 points) Determine if the statement is true or false. Justify your answer! (credit will be given only if a valid justification is provided).

   (a) If A is a $n \times n$ matrix and there is a vector $\vec{b}$ in $\mathbb{R}^n$, such that the equation $A\vec{x} = \vec{b}$ is inconsistent, then the equation $A\vec{x} = \vec{0}$ has a non-zero solution.

   (b) If the set $\{v_1, v_2, v_3\}$ of three vectors in $\mathbb{R}^5$ is linearly dependent, then so is the set $\{T(v_1), T(v_2), T(v_3)\}$, for every linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^6$.

   (c) If A is a $m \times n$ matrix and $n > m$, then the matrix equation $A\vec{x} = \vec{b}$ has infinitely many solutions for every vector $\vec{b}$ in $\mathbb{R}^m$.

5. (20 points) Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ a) Find the general form of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which satisfies $AB = BA$. Justify your answer.

   b) Let A and B be square $n \times n$ matrices, such that the columns of B are linearly independent. Assume that the equation $(AB)\vec{x} = \vec{0}$ has a non-zero solution $\vec{x}$. Show that the columns of A are linearly dependent. Hint: $(AB)\vec{x} = A(B\vec{x})$.