

$$\begin{pmatrix} + & - \\ - & + \\ + & - \end{pmatrix}$$

1. (20 points)

(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 5 & 0 & 4 \\ -2 & 3 & -4 \\ 2 & 0 & 7 \end{pmatrix}$

$$is -(\lambda - 3)^2(\lambda - 9).$$

cofactor expansion along 2nd-column

$$\det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & 0 & 4 \\ -2 & 3-\lambda & -4 \\ 2 & 0 & 7-\lambda \end{pmatrix} \stackrel{\downarrow}{=} +(-\lambda) \det \begin{pmatrix} 5-\lambda & 4 \\ 2 & 7-\lambda \end{pmatrix} =$$

$$= (3-\lambda) ((5-\lambda)(7-\lambda) - 8) = (3-\lambda)(\lambda-3)(\lambda-9) = -(\lambda-3)^2(\lambda-9)$$

$$\lambda^2 - 12\lambda + 27$$

(b) Find a basis of \mathbb{R}^3 consisting of eigenvectors of A .

Basis for 3-eigenspace (for $\text{null}(A-3I)$)

$$A-3I = \begin{pmatrix} 2 & 0 & 4 \\ -2 & 0 & -4 \\ 2 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\{v_1, v_2\}$ is a basis for the 3-eigenspace

Basis for 9-eigenspace (for $\text{null}(A-9I)$)

$$A-9I = \begin{pmatrix} -4 & 0 & 4 \\ -2 & -6 & -4 \\ 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\{v_3\}$ is a basis for the 9-eigenspace

- (c) Find an invertible matrix P and a diagonal matrix D such that the matrix A above satisfies

$$P^{-1}AP = D$$

$$P = (v_1 \ v_2 \ v_3) = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

The i -th diagonal entry is the eigenvalue of v_i .

- (d) Let B be a 5×5 matrix with characteristic polynomial

$-(\lambda - 1)^2(\lambda - 2)(\lambda - 3)(\lambda - 4)$. Assume that the rank of $B - I$ is 3. Is B necessarily diagonalizable? Justify your answer.

Recall that The algebraic multiplicity of an eigenvalue λ_i is the maximal power of $(\lambda - \lambda_i)$ that divides the characteristic polynomial.

The geometric multiplicity of an eigenvalue λ_i is the dimension of the λ_i -eigenspace $\text{Null}(A - \lambda_i I)$.

Theorem: A square matrix is diagonalizable (there exists a real matrix P , such that $P^{-1}AP$ is diagonal) if and only if (1) its characteristic poly factors as a product of linear terms, and

(2) The geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

In general, we have for each eigenvalue λ_i

$$1 \leq \text{geometric mult.} \leq \text{algebraic mult.}$$

The algebraic multiplicity of the eigenvalues 2, 3, 4 is 1, hence equal to its geometric multiplicity. Geometric mult. of 1 = $\dim(\text{Null}(B - 1I)) = 5 - \text{rank}(B - I) = 5 - 3 = 2$, which is the algebraic multiplicity of 1. Item no. the Rank nullity theorem DTAGONALIZABLE.

2. (20 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .6 & .4 \\ .3 & .7 \end{pmatrix}$. $= \frac{1}{10} \begin{pmatrix} 6 & 4 \\ 3 & 7 \end{pmatrix}$

(a) The eigenvalue of v_1 is 1

$$\frac{1}{10} \begin{pmatrix} 6 & 4 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigenvalue of v_2 is $\frac{3}{10}$

$$\frac{1}{10} \begin{pmatrix} 6 & 4 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 12 \\ -9 \end{pmatrix} = \left(\frac{3}{10}\right) \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

$$(v_1 | v_2 | \begin{pmatrix} 1 \\ 8 \end{pmatrix}) = \left(\begin{array}{cc|c} 1 & 4 & 1 \\ 1 & -3 & 8 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & -7 & 7 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array} \right)$$

$$\begin{pmatrix} 1 \\ 8 \end{pmatrix} = 5 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v_1} + (-1) \underbrace{\begin{pmatrix} 4 \\ -3 \end{pmatrix}}_{v_2} \quad \left[\begin{pmatrix} 1 \\ 8 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

(c) Compute $A^{20} \begin{pmatrix} 1 \\ 8 \end{pmatrix}$.

$$A^{20} \left(5v_1 + (-1)v_2 \right) = 5 \underbrace{A^{20}(v_1)}_{1^{20} \cdot v_1} - \underbrace{A^{20}(v_2)}_{\left(\frac{3}{10}\right)^{20} v_2} =$$

$$= 5v_1 - \left(\frac{3}{10}\right)^{20} v_2 = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left(\frac{3}{10}\right)^{20} \begin{pmatrix} 4 \\ -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 5 - 4 \left(\frac{3}{10}\right)^{20} \\ 5 + 3 \left(\frac{3}{10}\right)^{20} \end{pmatrix}$$

(d) As n gets larger, the vector $A^n \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ approaches $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$. Justify your answer.

$$\lim_{n \rightarrow \infty} A^n \begin{pmatrix} 1 \\ 8 \end{pmatrix} = \lim_{n \rightarrow \infty} \left(5 \underbrace{A^n(v_1)}_{5v_1} - \underbrace{A^n(v_2)}_{v_1} \right) = \lim_{n \rightarrow \infty} 5v_1 - \underbrace{\left(\frac{3}{10}\right)^n v_2}_{\substack{\downarrow \\ 0}} = 5v_1 = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

(e) Let B be an invertible $n \times n$ matrix and v an eigenvector of B with eigenvalue 5. Show that v is an eigenvector of the inverse matrix B^{-1} as well and compute its eigenvalue.

v is an eigenvector of B with eigenvalue 5

$$\Leftrightarrow Bv = 5v \quad (\text{Multiply both sides by } B^{-1})$$

$$\Leftrightarrow v = 5B^{-1}v$$

$$\Leftrightarrow B^{-1}(v) = \frac{1}{5}v$$

v is an eigenvector of B^{-1} with eigenvalue $\frac{1}{5}$.

3. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find the projection of b to the plane $\text{Col}(A)$ spanned by the columns of A .

The columns $\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{a}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ are orthogonal, hence an orthogonal basis for $W = \text{Col}(A)$. We can thus use the formula:

$$\text{Proj}_W(b) = \left(\frac{\vec{b} \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1} \right) \vec{a}_1 + \left(\frac{\vec{b} \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2} \right) \vec{a}_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} =$$

$$= \underbrace{\left(\frac{6}{6} \right)}_{1} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \underbrace{\left(\frac{3}{3} \right)}_{1} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

(b) Find the distance from b to $\text{Col}(A)$.

$$\text{dist}(\vec{b}, \underbrace{\text{Col}(A)}_W) = \|\vec{b} - \text{Proj}_W(\vec{b})\| = \left\| \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \right\| =$$

$$= \sqrt{2^2 + (-2)^2 + 0^2} = \sqrt{8}$$

(c) Find a vector x in \mathbb{R}^2 , for which the distance $\|Ax - b\|$ from Ax to b is equal to the distance from b to $\text{Col}(A)$. Hint: The vector Ax is in $\text{Col}(A)$ for every vector x .

Need $\|A\vec{x} - \vec{b}\| = \|\text{Proj}_W(\vec{b}) - \vec{b}\|$. So take \vec{x} to be the

solution of $A\vec{x} = \text{Proj}_W(\vec{b}) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 = 1 \\ x_2 = 1 \end{array}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

check: $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \text{Proj}_W(b)$.

4. (20 points) Consider the following orthogonal basis of \mathbb{R}^3

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Find the coordinates of the vector $b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in the above basis.

$$\vec{b} = c_1 v_1 + c_2 v_2 + c_3 v_3, \text{ where } c_i = \frac{\vec{b} \cdot v_i}{\|v_i\|^2}$$

$$c_1 = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} = \frac{2}{6} = \frac{1}{3}$$

$$c_2 = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} = \frac{2}{2} = 1, \quad c_3 = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} = \frac{-1}{3} = -\frac{1}{3}$$

(b) Normalize the above basis $\{v_1, v_2, v_3\}$ to an orthonormal basis $\{u_1, u_2, u_3\}$.

$$u_i = \frac{1}{\|v_i\|} v_i : \text{ so } u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, u_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(c) Let A be an $n \times n$ orthogonal matrix and v an eigenvector of A in \mathbb{R}^n . Show that the eigenvalue of v is either 1 or -1. Hint: Consider the length of Av .

As A is orthogonal, we have $(A\vec{v}) \cdot (A\vec{w}) \stackrel{\text{dot product}}{\equiv} \vec{v} \cdot \vec{w}$ for every two vectors \vec{v}, \vec{w} in \mathbb{R}^n .

Taking $\vec{v} = \vec{w}$ and letting λ be the eigenvalue of \vec{v} , we get:

$$A\vec{v} \cdot A\vec{v} = (\lambda\vec{v}) \cdot (\lambda\vec{v}) = \lambda^2 \vec{v} \cdot \vec{v}$$

$$\stackrel{\text{(*)}}{\Rightarrow} \vec{v} \cdot \vec{v} \neq 0$$

Hence, $\lambda^2 = 1$. So $\lambda = 1$ or $\lambda = -1$.

5. (20 points)

- (a) If the null space of an 8×5 matrix is 2 dimensional, what is the dimension of the row space of A ? Justify your answer.

$$\dim(\text{Row}(A)) = \text{rank}(A) \stackrel{\substack{\uparrow \\ \text{Rank-Nullity Theorem}}}{=} 5 - \dim(\text{Null}(A)) = 5 - 2 = 3.$$

$$\tilde{B} \quad \begin{matrix} f \\ g \\ h \end{matrix}$$

- (b) Show that the first three Laguerre polynomials $\{1, 1-t, 2-4t+t^2\}$ form a basis of \mathbb{P}_2 . Explain, in complete sentences, why it is linearly independent and why it spans \mathbb{P}_2 .

Let $B = \{1, t, t^2\}$ be the standard basis of \mathbb{P}_2 .

The coordinate linear transformation $[I_B : \mathbb{P}_2] \rightarrow \mathbb{R}^3$ is one-to-one and onto (an isomorphism) hence, the set of vectors \tilde{B} is linearly independent if and only if the coordinate vectors $\{\begin{bmatrix} f \\ B \end{bmatrix}, \begin{bmatrix} g \\ B \end{bmatrix}, \begin{bmatrix} h \\ B \end{bmatrix}\}$ in \mathbb{R}^3 is linearly independent, and \tilde{B} spans \mathbb{R}^3 if and only if Σ spans \mathbb{R}^3 .

The matrix

$$A = \left(\begin{bmatrix} f \\ B \end{bmatrix} \begin{bmatrix} g \\ B \end{bmatrix} \begin{bmatrix} h \\ B \end{bmatrix} \right) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \text{ has a pivot in}$$

every column, hence Σ is linearly independent, and a pivot in every row, hence Σ spans the whole of \mathbb{R}^3 . Thus \tilde{B} is linearly independent in \mathbb{P}_2 and spans the whole of \mathbb{P}_2 .

- v_1 v_2
- (c) Let \mathcal{B} be the basis $\left\{ \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\}$ of \mathbb{R}^2 and $[]_{\mathcal{B}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the coordinate linear transformation sending a vector v to its coordinate vector $[v]_{\mathcal{B}}$ relative to the basis \mathcal{B} . Find the matrix A of the linear transformation $[]_{\mathcal{B}}$.
Justify your answer! Hint: Multiplication by A should transform a vector v into its coordinate vector $[v]_{\mathcal{B}}$.

$Av = [v]_{\mathcal{B}}$ for every vector in \mathbb{R}^2 .

Let $[]_{\mathcal{B}}^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the inverse linear trans

Then $[]_{\mathcal{B}}^{-1}(Av) = []_{\mathcal{B}}^{-1}([v]_{\mathcal{B}}) = v$.

So, the matrix of $[]_{\mathcal{B}}^{-1}$ is A^{-1} .

Now, the matrix of $[]_{\mathcal{B}}^{-1}$ is $P = (v_1 \ v_2)$

since it takes $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ to $c_1 v_1 + c_2 v_2$.

$$\text{So } A = P^{-1} = (v_1 \ v_2)^{-1} = \begin{pmatrix} 1 & -2 \\ -4 & 9 \end{pmatrix}^{-1} = \frac{1}{9-8} \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix},$$