Midterm 1

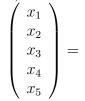
Fall 2008

Name:\_\_\_\_\_

1. (15 points) a) Show that the row **reduced** echelon form of the augmented matrix  $x_1 + x_3 - x_4 - 2x_5 = 2$ of the system  $x_1 + x_2 + 3x_3 = 1$  is  $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -1 \end{pmatrix}$ . Use at most five elementary operations. Show all your work. Clearly write in words each

elementary row operation you used.

b) Find the general solution for the system.



2. (20 points) You are given that the row reduced echelon form of the matrix

$$A = \begin{pmatrix} 3 & 6 & 1 & 2 & 6 & -4 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 1 & 2 & 0 & 0 & 1 & -1 \\ 1 & 2 & 2 & 0 & -1 & 1 \end{pmatrix}$$
 is  $B = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ . You do **not**

need to verify this statement.

(a) Write the general solutions of the system  $A\vec{x} = \vec{0}$  in parametric form  $\vec{x} = (\text{first free variable})\vec{v}_1 + (\text{second free variable})\vec{v}_2 + \dots$ 

(b) Let  $T : \mathbb{R}^6 \to \mathbb{R}^4$  be the linear transformations given by  $T(\vec{x}) = A\vec{x}$ . Find a finite set of vectors in ker(T), which spans ker(T). Explain why the set you found spans ker(T).

(c) Let  $\vec{a}_j$  be the *j*-th column of *A*. Explain, without any further calculations, why  $\vec{a}_6$  belongs to span $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$ .

(d) Is the image of T equal to the whole of  $\mathbb{R}^4?$  Justify your answer.

3. (a) (10 points) Determine for which values of k is the matrix  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & k+1 \end{pmatrix}$  invertible, and compute the inverse, when it exists.

(b) (2 points) Check that the matrix you found is indeed  $A^{-1}$ .

(c) (8 points) Let A, B, C be  $n \times n$  matrices, with A invertible, which satisfy the equation  $ACA^{-1} - A = B$ . Express C in terms of A and B. Show all your work.

- 4. (20 points) Let  $T : \mathbb{R}^3 \to \mathbb{R}^2$  be a linear transformation with standard matrix A. Assume that the image of T is the whole of  $\mathbb{R}^2$ . Carfully **justify** your answers to the following questions.
  - (a) The rank of A is: \_\_\_\_.

(b) Describe geometrically the kernel of T.

(c) Consider the standard matrix A of T. Fix one solution  $\vec{p}$  of the equation

$$A\vec{x} = \left(\begin{array}{c}1\\2\end{array}\right).$$

Show that if a vector  $\vec{x}$  is a solution of the above equation, then  $\vec{x} - \vec{p}$  belongs to the kernel of T. Show also the converse: If  $\vec{x} - \vec{p}$  belongs to the kernel of T, then the vector  $\vec{x}$  is a solution of the above equation.

- 5. (25 points) Let L be the line in  $\mathbb{R}^2$  through the origin and the non-zero vector  $\vec{u}$ . Recall that the projection  $Proj_L : \mathbb{R}^2 \to \mathbb{R}^2$  of the plane onto the line L is given by the formula  $Proj_L(\vec{x}) = \left(\frac{\vec{u} \cdot \vec{x}}{\vec{u} \cdot \vec{u}}\right) \vec{u}$ .
  - (a) Use the algebraic properties of the dot product to show that  $Proj_L$  is a linear transformation. In other words, verify the following identities, for any two vectors  $\vec{v}, \vec{w}$  and for every scalar k.
    - i.  $Proj_L(\vec{v} + \vec{w}) = Proj_L(\vec{v}) + Proj_L(\vec{w}).$

ii.  $Proj_L(k\vec{v}) = kProj_L(\vec{v}).$ 

(b) Let u = (1, 1). Use the above formula for  $Proj_L(\vec{x})$  to find the standard matrix P of  $Proj_L$ .

(c) Find the matrix R of the rotation of the plane 90 degrees counterclockwise.

(d) Let  $T : \mathbb{R}^2 \to \mathbb{R}^2$  be the linear transformation, which first rotates a vector 90 degrees counterclockwise, and then projects the resulting vector onto the line L. Express the standard matrix A of T in terms of the standard matrices P of  $Proj_L$  and R of the rotation: A =\_\_\_\_\_. Use this expression to compute A.

(e) (5 **bonus** points) Find a vector  $\vec{v}$  in  $\mathbb{R}^2$ , such that the linear transformation T in part 5d admits the new description  $T(\vec{x}) = R(Proj_{\tilde{L}}(\vec{x}))$ , where  $\tilde{L}$  is the line through the origin and  $\vec{v}$ . Justify your answer.