# Homework 6

**Exercise 1:** Suppose that you run the Monte-Carlo algorithm to compute $\pi$ 10'000 times and observe 7932 points inside the circle. What is your estimation for the value of $\pi$? Using Chebyshev describe how accurate your estimation of $\pi$, the answer should be in the form should be in the form: based on my computation the number $\pi$ belong to the interval $[a, b]$ with probability .95.

**Exercise 2:** Suppose you start with a fortune of 1 and throw a coin $n$ times. A very rich and not very bright friend of yours pays you $\frac{11}{5}k$ for a bet of $k$ if the throw is heads and nothing if the throw is tail.

1. Suppose you bet everything on each throw. What is your expected gain after 20 throws? What is the probability that you win nothing after 20 throws? What is the variance of your gain after 20 throws?
2. What does Kelly's formula suggest you should do? What is your expected gain and its variance after 20 throws in that case?

**Exercise 3:** Consider the proportional play strategy, but now every single bet of 1 unit leads to three possible outcomes

$$P(\text{ Win } 4) = 1/2 \quad P(\text{ Lose } 1) = 1/4 \quad P(\text{ Lose } 4) = 1/4 .$$

What is the optimal proportion $f^*$ of your fortune you should invest?

**Exercise 4: (Kelly betting with interest)**.

1. You are given the possibility to make a risky bet which returns with probability $p$, \$ $\gamma$ for every dollar you bet. On the other hand if you do not bet you can invest your money in a bank and earn during the period of the bet an interest rate $\alpha$. You decide to follow the proportional play strategy and invest a portion $f$ of your money in the bet and to invest the remaining portion $(1 - f)$ in the bank.

   Find the optimal $f$ which will maximize the growth rate of your fortune.
2. Suppose you are faced with a 100% safe investment returning 5% or a 90% safe investment returning 25%. Calculate how to invest your money using the Kelly strategy. Calculate the effective rate of return on your investment over the long term.

**Exercise 5:** A particular game pays $\gamma_1$ times the amount staked with a probability $p$ and $\gamma_2$ times the amount staked with a probability $1-p$ and we will take take

$$\gamma_1 > 1, \quad 0 \le \gamma_2 < 1 \text{ and } p\gamma_1 + (1-p)\gamma_2 > 1$$

1. Suppose your initial fortune is $X_0$, and you follow a proportional play strategy where you invest a proportion $f$ of your fortune in the bet and keep a proportion $(1-f)$. After $n$ bets your fortune will take the form $X_n = Q_n \cdots Q_1 X_0$ where $Q_i$ are IID random variables. Describe the random variable $Q_i$?

2. If you optimize your growth rate, show that in this case, the optimal Kelly fraction is
$$f^* = \min \left\{ \frac{p\gamma_1 + (1-p)\gamma_2 - 1}{(\gamma_1 - 1)(1-\gamma_2)}, 1 \right\}$$

**Exercise 6:** Suppose you are an investor in technology companies and you want to make quick money. You see thousands of new companies being brought to the stock market every year. It is very hard to predict in which direction the stock prices will move but for half of the companies the stock prize will rise 80% during the first day after the stock ins introduced while half of the other companies the stock price will fall 60 during this period. Your investment strategy is to buy such stock and sell them at the end of the day. Your initial fortune is $10,000.

1. Suppose that for 1000 days you pick every day one stock at random and invest all your fortune in it. Estimate, using for example Chebyshev equality, that you get more than 55 % of the time stocks whose prizes rise. If you assume that you will not pick more than 55% of the time stocks whose prize rises, what is the best scenario for your fortune after 500 days?

2. Use now instead a Kelly strategy where you invest a fixed proportion of your fortune every time. Using the result of Exercise 3, what is the optimal Kelly fraction? What is the growth rate in this case?