MATH 331.1, Fall 2007 : Quizz #5 solution

Your Name: ______________________________________

The quizz has 1 question worth 10 points.

(a) (2.5 pts). Find the *general solution* for the second order system $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 12y = 0$.

Solution: Try a solution of the form $y(t) = e^{st}$. Inserting into the equation gives

$$s^2 + 7s + 12 = 0 .$$

which has solutions

$$s = \frac{-7 \pm \sqrt{49 - 48}}{2} = -3, -4 .$$

The general solution is then

$$y(t) = k_1 e^{-3t} + k_2 e^{-4t} .$$

(b) (2.5 pts). Using complex numbers find a complex *particular solution* y_c of the second order system $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 12y = 8e^{4it}$.

Solution: Try a solution of the form $y(t) = ae^{4it}$. Inserting into the equations give

$$(4i)^2 ae^{4it} + 7(4i)ae^{4it} + 12ae^{4it} = 8e^{4it} .$$

which gives

$$a(-16 + 28i + 12) = 8 \qquad \text{or} \qquad a = \frac{8}{-4 + 8i} = \frac{2}{-1 + 7i}$$

and

$$a = \frac{2}{-1 + 7i} \frac{(-1 - 7i)}{(-1 - 7i)} = \frac{-2 - 14i}{1 + 49} = -\frac{1}{25} - i\frac{7}{25}$$

The complex particular solution is then

$$y_c(t) = \left(-\frac{1}{25} - i\frac{7}{25}\right) e^{4it} .$$

(c) (2.5 pts). Find the a real *particular solution* y_p for the second order system $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 12y = 8\sin(4t)$.

Solution: To find the particular solution we need to find the *imaginary part* of $y_c(t)$ (imaginary part because we have a force term $8\sin(4t)$.)

We have

$$\begin{aligned} y_c(t) &= (-1/25 - i7/25)e^{4it} = (-1/25 - i7/25)(\cos(4t) + i\sin(4t)) \\ &= [-1/25\cos(4t) + 7/25\sin(4t)] + i\,[-7/25\cos(4t) - 1/25\sin(4t)] \end{aligned}$$

Therefore

$$y_p(t) = -7/25\cos(4t) - 1/25\sin(4t)\,.$$

(d) (2.5 pts). Find the *amplitude* and the phase angle of the particular solution y_p.

To find the amplitude we rewrite a found in (b) as

$$a = -1/25 - i7/25 = |a|e^{i\theta}$$

with

$$\begin{aligned} |a| &= \sqrt{(-1/25)^2 + (-7/25)^2} = \sqrt{2/25} \\ \theta &= \arctan\left(\frac{-7/25}{-1/25}\right) = \arctan(-7)\,. \end{aligned}$$

The solution $y_c(t)$ is then

$$Y_c(t) = \sqrt{2/25}e^{i\theta}e^{i4t} = \sqrt{2/25}e^{i\theta+4t} = \sqrt{2/25}\,(\cos(4t+\theta) + i\sin(4t+\theta))$$

The particular solution $y_p(t)$ is then

$$\sqrt{2/25}\sin(4t+\theta) = \sqrt{2/25}\cos(4t+\theta-90^o)$$

The *amplitude* is $\sqrt{2/25}$ and the *phase angle* is $\arctan(-7) - 90^o$.