MATH 331.1, Fall 2007 : Quizz #4 solution

Your Name: ____

The quizz has 2 question worth 5 points each.

For the following 2 linear systems of the form $\dfrac{d\mathbf{Y}}{dt} = A\mathbf{Y}$,
(a) Find the eigenvalues and determine the *type* of the system (source, sink, saddle, center, spiral source, spiral sink, degenerate eigenvalue).

(b) If the eigenvalues are *real* compute the *eigenvectors* and draw the *phase portrait* of the system in the x, y-plane.
If the eigenvalue are *complex* determine the *directions of oscillations* (clockwise or counterclockwise) and draw the *phase portrait* of the system in the x, y-plane.

1. $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

Solution:
1(a): Eigenvalues and type.
The equation for the eigenvalues is

$$0 = \det \begin{pmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 2 = \lambda^2 - 2\lambda + 3\,.$$

We have

$$\lambda = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm i\sqrt{2}$$

The eigenvalue are *complex* and the real part is $+1$ so positive. The type is thus a *spiral source.*

In order to determine the *directions of oscillations* pick a random point, say $(1, 0)$. We have

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1/4 \\ 1/4 & 1-\lambda \end{pmatrix}.$$

So at the point $(1, 0)$ i.e., on the x-axis the vector field point in the $(1, -1)$ direction, i.e., in the southeast direction. Therefore the solution spiral in the *clockwise* direction.

1(b) Phase portrait.

2) $\quad A = \begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix}$

Solution:

2(a): Eigenvalues and type.
The equation for the eigenvalues is

$$0 \;=\; \det\begin{pmatrix} 1-\lambda & 1/4 \\ 1/4 & 1-\lambda \end{pmatrix} \;=\; (1-\lambda)^2 - 1/16 \;=\; \lambda^2 - 2\lambda + 15/16\,.$$

We have then

$$\lambda \;=\; \frac{2 \pm \sqrt{4 - 4\times 15/16}}{2} \;=\; 1 \pm \frac{1}{4} \;=\; 3/4, 5/4$$

Therefore we have a *source*.
The eigenvector equations are
(a) $\lambda = 3/4$

$$\begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \;=\; 3/4\begin{pmatrix} x \\ y \end{pmatrix}$$

which gives the equation $x + y/4 = 3x/4$ or $x + y = 0$. For $\lambda = 3/4$ the eigenvector is $(1, -1)$.
(b) $\lambda = 5/4$

$$\begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \;=\; 5/4\begin{pmatrix} x \\ y \end{pmatrix}$$

which gives $x + y/4 = 5x/4$ or $x - y = 0$. For $\lambda = 5/4$ the eigenvector is $(1, 1)$.

2(b) Phase portrait.