MATH 331.1, Fall 2007 : Quizz #1 Solution

Your Name: ____

The quizz has 2 questions worth 5 points each.

1. Solve the initial value problem $\dfrac{dy}{dt} = \dfrac{1+y^2}{y}\,, \quad y(0) = -2\,.$

Solution: The equation is separable and we obtain

$$\int \frac{y}{1+y^2}dy = \int dt$$

Using the change of variables $u = 1 + y^2$, $du = 2ydy$ we have

$$\int \frac{y}{1+y^2}dy \;=\; \frac{1}{2}\int \frac{1}{u}du \;=\; \frac{1}{2}\ln(|u|) \;=\; \frac{1}{2}\ln(1+y^2)$$

So we obtain

$$\frac{1}{2}\ln(1+y^2) \;=\; t + C$$

or with $K = e^{2C}$

$$1 + y^2 \;=\; Ke^{2t}$$

The general solution is then

$$y \;=\; \pm\sqrt{Ke^{2t} - 1}\,.$$

The initial condition $y(0) = -2$ gives the equation

$$-2 \;=\; \pm\sqrt{K-1}$$

This tells us that we should choose $K = 5$ **and** the negative sign in front of the square root. The solution is then

$$y(t) \;=\; -\sqrt{5e^{2t} - 1}\,.$$

2. Consider the differential equation $\dfrac{dy}{dt} = 3y^3 - 12y$

(a) Find the equilibrium points, draw the phase line, and identify the equilibrium points as source, sink, or node.

Solution: The equilibrium points are determined by the equation

$$3y^3 - 12y \;=\; 3y(y^2 - 4) \;=\; 0$$

There are 3 equilibrium points $y = 0$, $y = 2$, and $y = -2$. The function $f(y) = 3y^3 - 12y^2$ has the following sign

$$\begin{aligned} y < -2 \quad &: \quad f(y) < 0 \\ -2 < y < 0 \quad &: \quad f(y) > 0 \\ 0 < y < -2 \quad &: \quad f(y) < 0 \\ y > 2 \quad &: \quad f(y) > 0 \end{aligned} \tag{1}$$

This implies that

- -2 is a source.
- 0 is a sink.
- 2 is a source.

(b) Sketch the solutions with initial conditions $y(0) = 2$, $y(0) = -1$, $y(0) = -3$.

Solution:

- If $y(0) = 2$ then $y(t) = 2$ for all t since 2 is an equilibrium point.
- If $y(0) = -1$ then, since $-2 < -1 < 0$ the solution is increasing since $f(-1) > 0$. As $t \to \infty$ $y(t) \to 0$ and as $t \to -\infty$ $y(t) \to -2$.
- If $y(0) = -1$ then, since $-3 < -2$ the solution is decreasing increasing since $f(-3) < 0$. For positive t, $y(t) \to -\infty$ and as $t \to -\infty$ $y(t) \to -2$.