# Math 331.1: Review problems

**Exercise 1** Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where $A$ is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \quad (c)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (d)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (e)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

In each case of the five cases

1. Determine the type of the system, i.e., sink, source, saddle, center, spiral source, spiral sink, center, degenerate eigenvalues.
2. Draw the phase portrait of the system. If the eigenvalue are real you need to compute the eigenvectors and indicate them clearly on the phase portrait. If the eigenvalues are complex you need to determine the orientations of the oscillations (clockwise or counterclockwise).
3. Draw a rough graph of a typical solution $x(t)$, $y(t)$. Note that you *do not* need to solve the system to do this! If the eigenvalues are complex indicate clearly in your graph the period of the oscillations.

**Exercise 2** Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where $A$ is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \quad (c)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (d)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (e)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

In each case

1. Compute the *general solution* of $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$.
2. Solve the *initial value problem* $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$, $\mathbf{Y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

**Exercise 3** Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where $A$ is given by

$$(a)\begin{pmatrix} a & a \\ a & 1 \end{pmatrix} \quad (b)\begin{pmatrix} 1 & 0 \\ a & -4 \end{pmatrix}$$

and $a$ is a parameter. Use the trace-determinant plane to determine the different types of the systems and the bifurcations of the systems as the parameter $a$ increases on the real line.

**Exercise 4** Consider the second order equation

$$4\frac{d^2y}{dt^2} + k\frac{dy}{dt} + 5y = 0$$

where $k$ is a parameter with $-\infty < k < \infty$. As $k$ varies describe (using the trace-determinant plane) the different types of the systems and the bifurcations.

**Exercise 5** Compute the general solution for the

$$\begin{aligned}
(a) \quad \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y &= e^{-4t} \\
(b) \quad \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y &= e^{2t} \\
(c) \quad \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y &= e^{t} + e^{-2t} \\
(d) \quad \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y &= t^2 + 1
\end{aligned}$$

**Exercise 6** Consider the equation

$$\frac{dy^2}{dt^2} + 4\frac{dy}{dt} + 7y = 6\sin(3t)\,.$$

1. Find the general solution.
2. Describe the behavior of the general solution as $t \to \infty$ and graph a typical solution.
3. Compute the amplitude and phase angle.

**Exercise 7** Solve the initial value problem

$$\frac{dy^2}{dt^2} - 4\frac{dy}{dt} - 5y = 6\sin(3t)\,,\ y(0) = 2, y'(0) = -1\,.$$

**Exercise 8** Consider the equation

$$\frac{dy^2}{dt^2} + 8y = 6\sin(3t)\,.$$

1. Determine the frequency of the beating.
2. Determine the frequency of the rapid oscillations.
3. Give a rough sketch of typical solution indicating clearly the results obtained in 1. and 2.

**Remark**: To answer this questions you do not need to compute the solutions explicitly.

**Exercise 9** Find the general solution of

$$(a)\ \frac{dy^2}{dt^2} + 16y = 3\sin(4t)\,.$$

$$(b)\ \frac{dy^2}{dt^2} + 16y = 5\cos(2t)$$

**Exercise 10** Solve the initial value problem

$$(a)\ \frac{dy^2}{dt^2} + 16y = 3\sin(4t)\,,\ y(0) = 1, y'(0) = 0$$

$$(b)\ \frac{dy^2}{dt^2} + 16y = 5\cos(2t),\ y(0) = 2, y'(0) = 2$$

**Exercise 11** Compute the inverse Laplace transform of the following functions

$$(a)\ \frac{7}{s+2} \quad (b)\ \frac{e^{-8s}}{s+2} \quad (c)\ \frac{e^{-5s}}{s^2+2s-2} \quad (d)\ \frac{1}{s^2+2s+2} \quad (e)\ \frac{e^{-2s}}{s^2+2s+2}$$

$$(g)\ \frac{2s-5}{s^2+2s+2} \quad (h)\ \frac{e^{-3s}}{(s^2+1)(s^2+4)} \quad (i)\ \frac{e^{-2s}}{(s-1)(s^2+4s+5)}$$

**Exercise 12** The function $h(t)$ is given by

$$h(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1 \\ 2 & \text{if} \quad 1 \le t < 3 \\ 0 & \text{if} \quad 3 \le t\,. \end{cases}$$

1. Compute the Laplace transform of $h(t)$. *Hint:* Write $h$ as a combination of $u_a(t)$ for suitable $a$'s.
2. Solve the equation $\frac{dy}{dt} + 3y = h(t)$.

**Exercise 13** Use the Laplace transform method to solve the following initial value problems.

1. $\frac{dy}{dt} + 5y = 5u_2(t)\,,\ y(0) = -7\,.$ Make also a graph of the solutions.
2. $\frac{dy}{dt} + 4y = -3u_4(t)e^{2(t-4)}\,,\ y(0) = 2\,.$ What is $\lim_{t\to\infty} y(t)$?
3. $\frac{dy^2}{dt^2} + 4y = 2u_2(t)\cos(3(t-2)) \quad y(0) = 0,\ y'(0) = 1\,.$
4. $\frac{dy^2}{dt^2} + 4y = 3u_1(t)e^{-(t-1)} \quad y(0) = 0,\ y'(0) = 1\,.$ How does the solution behave for large $t$?
5. $\frac{dy^2}{dt^2} + 2\frac{dy}{dt} + 10y = u_4(t) \quad y(0) = 2,\ y'(0) = 0\,.$ What is $\lim_{t\to\infty} y(t)$? Make a graph of the solution.
6. $\frac{dy^2}{dt^2} + 5y = \delta_5(t) \quad y(0) = 2,\ y'(0) = 1\,.$ Make a graph of the solution.
7. $\frac{dy^2}{dt^2} + 4\frac{dy}{dt} + 7y = \delta_5(t) \quad y(0) = 6,\ y'(0) = -1\,.$ Make a graph of the solution.

**Exercise 14** For the following problems

1. Compute the Laplace transform of the solution.
2. Find the poles of the Laplace transform of the solution.
3. Discuss the behavior of solution (make a schematic graph) showing clearly the behavior for large $t$.

**Remark:** To answer these questions you do not need to compute the inverse Laplace transfrom, i.e. no partial fraction expansions are necessary!

(a) $\dfrac{dy^2}{dt^2} + 4\dfrac{dy}{dt} + 10y \; = \; u_4(t)\cos(3(t-4))$

(b) $\dfrac{dy^2}{dt^2} + 4\dfrac{dy}{dt} + 10y \; = \; u_4(t)e^{-2(t-4)}\cos(3(t-4))$

(c) $\dfrac{dy^2}{dt^2} + 4\dfrac{dy}{dt} + 10y \; = \; u_4(t)e^{2(t-4)}\cos(3(t-4))$

(c) $\dfrac{dy^2}{dt^2} + 3\dfrac{dy}{dt} - 10y \; = \; u_4(t)\cos(3(t-4))$

(d) $\dfrac{dy^2}{dt^2} + 3\dfrac{dy}{dt} - 10y \; = \; u_4(t)e^{-2(t-4)}\cos(3(t-4))$