Math 331.1: Review problems

Exercise 1 *Review exercise 19, chapter 3, p. 373*

Exercise 2 *Review exercise 21, chapter 3, p. 373*

Exercise 3 *Review exercise 23, chapter 4, p. 445*

Exercise 4 *Review exercise 11, chapter 6, p. 622*

Exercise 5 Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where A is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \quad (c)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (d)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (e)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

In each case of the five cases

1. Determine the type of the system, i.e., sink, source, saddle, center, spiral source, spiral sink, center, degenerate eigenvalues.
2. Draw the phase portrait of the system. If the eigenvalue are real you need to compute the eigenvectors and indicate them clearly on the phase portrait. If the eigenvalues are complex you need to determine the orientations of the oscillations (clockwise or counterclockwise).
3. Draw a rough graph of a typical solution $x(t)$, $y(t)$. Note that you *do not* need to solve the system to do this! If the eigenvalues are complex indicate clearly in your graph the period of the oscillations.

Exercise 6 Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where A is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \quad (c)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (d)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (e)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

In each case

1. Compute the *general solution* of $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$.
2. Solve the *initial value problem* $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$, $\mathbf{Y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Exercise 7 Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where A is given by

$$(a)\begin{pmatrix} 1 & 0 \\ a & -4 \end{pmatrix} \quad (b)\begin{pmatrix} 0 & 3a \\ 1 & a \end{pmatrix}$$

and a is a parameter. Use the trace-determinant plane to determine the different types of the systems and the bifurcations of the systems as the parameter a increases on the real line.

Exercise 8 Consider the second order equation

$$4\frac{d^2y}{dt^2} + k\frac{dy}{dt} + 5y \;=\; 0$$

where k is a parameter with $-\infty < k < \infty$. As k varies describe (using the trace-determinant plane) the different types of the systems and the bifurcations.

Exercise 9 Consider the second order equation

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} + 2ky \;=\; 0$$

where k is a parameter with $-\infty < k < \infty$. As k varies describe (using the trace-determinant plane) the different types of the systems and the bifurcations.

Exercise 10 Compute the general solution for the

$$\begin{aligned}
(a) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y \;=\; e^{-4t} \\
(b) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y \;=\; e^{2t} \\
(c) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y \;=\; e^{t} + e^{-2t} \\
(d) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y \;=\; t^2 + 1
\end{aligned}$$

Exercise 11 Consider the equation

$$\frac{dy^2}{dt^2} + 4\frac{dy}{dt} + 7y \;=\; 6\sin(3t)\,.$$

1. Find the general solution.
2. Describe the behavior of the general solution as $t \to \infty$ and graph a typical solution.
3. Compute the amplitude and phase angle.

Exercise 12 Solve the initial value problem

$$\frac{dy^2}{dt^2} - 4\frac{dy}{dt} - 5y \;=\; 6\sin(3t)\,,\;\; y(0) = 2, y'(0) = -1\,.$$

Exercise 13 Consider the equation

$$\frac{dy^2}{dt^2} + 8y \;=\; 6\sin(3t)\,.$$

1. Determine the frequency of the beating.
2. Determine the frequency of the rapid oscillations.

3. Give a rough sketch of typical solution indicating clearly the results obtained in 1. and 2.

Remark: To answer this questions you do not need to compute the solutions explicitly.

Exercise 14 Find the general solution of

$$(a)\ \frac{dy^2}{dt^2} + 16y \;=\; 3\sin(4t)\,.$$

$$(b)\ \frac{dy^2}{dt^2} + 16y \;=\; 5\cos(2t)$$

Exercise 15 Solve the initial value problem

$$(a)\ \frac{dy^2}{dt^2} + 16y \;=\; 3\sin(4t)\,,\ \ y(0) = 1, y'(0) = 0$$

$$(b)\ \frac{dy^2}{dt^2} + 16y \;=\; 5\cos(2t),\ \ y(0) = 2, y'(0) = 2$$

Exercise 16 Compute the inverse Laplace transform of the following functions

$$(a)\ \frac{7}{s+2} \quad (b)\ \frac{e^{-8s}}{s+2} \quad (c)\ \frac{e^{-5s}}{s^2+2s-2} \quad (d)\ \frac{1}{s^2+2s+2} \quad (e)\ \frac{e^{-2s}}{s^2+2s+2}$$

$$(g)\ \frac{2s-5}{s^2+2s+2} \quad (h)\ \frac{e^{-3s}}{(s^2+1)(s^2+4)} \quad (i)\frac{e^{-2s}}{(s-1)(s^2+4s+5)}$$

Exercise 17 The function $h(t)$ is given by

$$h(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 \le t < 1 \\ 2 & \text{if} \quad 1 \le t < 3 \\ 0 & \text{if} \quad 3 \le t\,. \end{cases}$$

1. Compute the Laplace transform of $h(t)$. *Hint:* Write h as a combination of $u_a(t)$ for suitable a's.

2. Solve the equation $\dfrac{dy}{dt} + 3y = h(t)$.

Exercise 18 Use the Laplace transform method to solve the following initial value problems.

1. $\dfrac{dy}{dt} + 5y \;=\; 5u_2(t)\,,\ y(0) = -7\,.$ Make also a graph of the solutions.

2. $\dfrac{dy}{dt} + 4y \;=\; -3u_4(t)e^{2(t-4)}\,,\ y(0) = 2\,.$ What is $\lim_{t\to\infty} y(t)$?

3. $\dfrac{dy^2}{dt^2} + 4y \;=\; 2u_2(t)\cos(3(t-2)) \quad y(0) = 0,\ y'(0) = 1\,.$

4. $\dfrac{dy^2}{dt^2} + 4y \;=\; 3u_1(t)e^{-(t-1)} \quad y(0) = 0,\ y'(0) = 1\,.$ How does the solution behave for large t?

5. $\dfrac{dy^2}{dt^2} + 2\dfrac{dy}{dt} + 10y \;=\; u_4(t) \quad y(0) = 2,\; y'(0) = 0$. What is $\lim_{t\to\infty} y(t)$? Make a graph of the solution.

6. $\dfrac{dy^2}{dt^2} + 5y \;=\; \delta_5(t) \quad y(0) = 2,\; y'(0) = 1$. Make a graph of the solution.

7. $\dfrac{dy^2}{dt^2} + 4\dfrac{dy}{dt} + 7y \;=\; \delta_5(t) \quad y(0) = 6,\; y'(0) = -1$. Make a graph of the solution.